

CÁLCULO DE LÍMITES

1. Sea la función definida por:

$$f(x) = \begin{cases} x & \text{si } x \geq 1 \\ 1 & \text{si } x < 1 \end{cases}$$

Calcular, si existe, $\lim_{x \rightarrow 1} f(x)$

Solución:

Para que exista el límite debe verificarse que $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x)$.

$$\left. \begin{array}{l} \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x = 1 \\ \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1 = 1 \end{array} \right\} \rightarrow \lim_{x \rightarrow 1} f(x) = 1$$

2. Sea la función definida a trozos:

$$f(x) = \begin{cases} \ln x & \text{si } x \geq 1 \\ e^{x-1} & \text{si } x < 1 \end{cases}$$

Demuestra que no existe $\lim_{x \rightarrow 1} f(x)$

Solución:

Para que exista el límite debe verificarse que $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x)$.

$$\left. \begin{array}{l} \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \ln x = \ln 1 = 0 \\ \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} e^{x-1} = e^0 = 1 \end{array} \right\} \rightarrow \lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x) \rightarrow \nexists \lim_{x \rightarrow 1} f(x) = 1$$

3. Dadas las funciones

$$f(x) = \frac{x^2 - 2x}{x^3 + x} \qquad g(x) = \frac{x^2 - 5}{1 + x}$$

Hallar:

$$\begin{array}{lll} \text{a) } \lim_{x \rightarrow \infty} f(x) & \text{b) } \lim_{x \rightarrow +\infty} g(x) & \text{c) } \lim_{x \rightarrow +\infty} f(x) \cdot g(x) \\ \text{d) } \lim_{x \rightarrow +\infty} f(x) + g(x) & \text{e) } \lim_{x \rightarrow +\infty} f(x) - g(x) & \text{f) } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} \end{array}$$

Solución:

$$\begin{array}{ll} \text{a) } \lim_{x \rightarrow \infty} f(x) = 0 & \text{b) } \lim_{x \rightarrow +\infty} g(x) = +\infty \\ \text{c) } \lim_{x \rightarrow +\infty} f(x) \cdot g(x) = 1 & \text{d) } \lim_{x \rightarrow +\infty} f(x) + g(x) = +\infty \\ \text{e) } \lim_{x \rightarrow +\infty} f(x) - g(x) = -\infty & \text{f) } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = 0 \end{array}$$

4. Hallar h y k sabiendo que se cumple:

$$\lim_{x \rightarrow +\infty} \left[kx + h - \frac{x^3 + 1}{x^2 + 1} \right] = 0$$

Solución:

$$\lim_{x \rightarrow +\infty} \left[kx + h - \frac{x^3 + 1}{x^2 + 1} \right] = \lim_{x \rightarrow +\infty} \left[\frac{kx^3 + hx^2 + kx + h - x^3 - 1}{x^2 + 1} \right] = \lim_{x \rightarrow +\infty} \left[\frac{(k-1)x^3 + hx^2 + kx + h - 1}{x^2 + 1} \right] = 0$$

Para que el límite valga cero debe verificarse que el grado del numerador sea menor que el del denominador, por tanto, imponemos que el numerador sea un polinomio de grado 1, es decir:

$$\circ k - 1 = 0 \rightarrow k = 1$$

$$\circ h = 0$$

7. Calcular los siguientes límites:

a) $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x^2 - 9}$

b) $\lim_{x \rightarrow 1} \frac{x^3 - 3x + 2}{x^2 - 4x + 3}$

c) $\lim_{x \rightarrow 2} \frac{x^3 - 12x + 16}{x^3 - 3x^2 + 4}$

d) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 2x}$

e) $\lim_{x \rightarrow 3} \frac{x^2 - 6x + 9}{x^2 - 7x + 12}$

f) $\lim_{x \rightarrow 1} \frac{x^3 - x}{2x^2 + x - 3}$

Solución:

a) $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x^2 - 9} = \frac{0}{0} = \lim_{x \rightarrow 3} \frac{(\cancel{x-3})(x^2 + 3x + 9)}{(\cancel{x-3})(x+3)} = \lim_{x \rightarrow 3} \frac{x^2 + 3x + 9}{x+3} = \frac{27}{6} = \frac{9}{2}$

b) $\lim_{x \rightarrow 1} \frac{x^3 - 3x + 2}{x^2 - 4x + 3} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{(\cancel{x-1})(x-2)}{(\cancel{x-1})(x-3)} = \lim_{x \rightarrow 1} \frac{x-2}{x-3} = \frac{1}{2}$

c) $\lim_{x \rightarrow 2} \frac{x^3 - 12x + 16}{x^3 - 3x^2 + 4} = \frac{0}{0} = \lim_{x \rightarrow 2} \frac{(\cancel{x-2})^2(x+4)}{(\cancel{x-2})^2(x+1)} = \lim_{x \rightarrow 2} \frac{x+4}{x+1} = \frac{6}{3} = 2$

d) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 2x} = \frac{0}{0} = \lim_{x \rightarrow 2} \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})x} = \lim_{x \rightarrow 2} \frac{x+2}{x} = \frac{4}{2} = 2$

e) $\lim_{x \rightarrow 3} \frac{x^2 - 6x + 9}{x^2 - 7x + 12} = \frac{0}{0} = \lim_{x \rightarrow 3} \frac{(\cancel{x-3})(x-3)}{(\cancel{x-3})(x-4)} = \lim_{x \rightarrow 3} \frac{x-3}{x-4} = \frac{0}{-1} = 0$

f) $\lim_{x \rightarrow 1} \frac{x^3 - x}{2x^2 + x - 3} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{x(\cancel{x-1})(x+1)}{(\cancel{x-1})(2x+3)} = \lim_{x \rightarrow 1} \frac{x(x+1)}{2x+3} = \frac{2}{5}$

8. Calcular los siguientes límites:

$$a) \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$$

$$b) \lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1}$$

$$c) \lim_{x \rightarrow a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}}$$

$$d) \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - \sqrt{x+1}}{x}$$

$$e) \lim_{x \rightarrow 1} \frac{\sqrt{x^3-1}}{\sqrt{x^2-1}}$$

$$f) \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}}$$

$$g) \lim_{x \rightarrow 0} \frac{3 - \sqrt{x+9}}{x}$$

$$h) \lim_{x \rightarrow 5} \frac{2x-10}{\sqrt{x+4}-3}$$

$$i) \lim_{x \rightarrow 1} \frac{\sqrt{x^3-2x+1}}{\sqrt{4x^2-x-3}}$$

Solución:

$$a) \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \frac{0}{0} = \lim_{x \rightarrow 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

$$b) \lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

$$c) \lim_{x \rightarrow a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}} = \frac{0}{0} = \lim_{x \rightarrow a} \frac{x^2 - a^2}{\sqrt{x} - \sqrt{a}} = \lim_{x \rightarrow a} \frac{(x-a)(x+a)(\sqrt{x} + \sqrt{a})}{x-a} = \lim_{x \rightarrow a} (x+a)(\sqrt{x} + \sqrt{a}) = 4a\sqrt{a}$$

$$d) \lim_{x \rightarrow 0} \frac{\sqrt{1-x} - \sqrt{x+1}}{x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{(1-x-x-1)}{x(\sqrt{1-x} + \sqrt{x+1})} = \lim_{x \rightarrow 0} \frac{-2}{(\sqrt{1-x} + \sqrt{x+1})} = -1$$

$$e) \lim_{x \rightarrow 1} \frac{\sqrt{x^3-1}}{\sqrt{x^2-1}} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{\sqrt{(x-1)(x^2+x+1)}}{\sqrt{(x-1)(x+1)}} = \lim_{x \rightarrow 1} \sqrt{\frac{x^2+x+1}{x+1}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

$$f) \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}} = \frac{0}{0} = \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{(x-2)(x+2)}} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{0}{2} = 0$$

$$g) \lim_{x \rightarrow 0} \frac{3 - \sqrt{x+9}}{x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{9-x-9}{x(3+\sqrt{x+9})} = \lim_{x \rightarrow 0} \frac{-1}{(3+\sqrt{x+9})} = -\frac{1}{6}$$

$$h) \lim_{x \rightarrow 5} \frac{2x-10}{\sqrt{x+4}-3} = \frac{0}{0} = \lim_{x \rightarrow 5} \frac{(2x-10)(\sqrt{x+4}+3)}{x+4-9} = \lim_{x \rightarrow 5} 2(\sqrt{x+4}+3) = 12$$

$$i) \lim_{x \rightarrow 1} \frac{\sqrt{x^3-2x+1}}{\sqrt{4x^2-x-3}} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{\sqrt{(x-1)(x^2+x-1)}}{\sqrt{(x-1)(4x+3)}} = \lim_{x \rightarrow 1} \sqrt{\frac{x^2+x-1}{4x+3}} = \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$$

9. Calcular los siguientes límites:

$$a) \lim_{x \rightarrow 2} \left(\frac{x+2}{x-1} - \frac{2x+4}{x^2-1} \right)$$

$$b) \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$$

$$c) \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{4x}{x^2-9} \right)$$

$$d) \lim_{x \rightarrow +\infty} (\sqrt{x^2+1} - x)$$

$$e) \lim_{x \rightarrow +\infty} (\sqrt{x^2-9} - \sqrt{x^2-6x})$$

$$f) \lim_{x \rightarrow +\infty} (\sqrt{x^2+x+1} - \sqrt{x^2-2x-1})$$

Solución:

$$a) \lim_{x \rightarrow 2} \left(\frac{x+2}{x-1} - \frac{2x+4}{x^2-1} \right) = \infty - \infty = \lim_{x \rightarrow 2} \frac{x^2+3x+2-2x-4}{x^2-1} = \lim_{x \rightarrow 2} \frac{x^2+x-2}{x^2-1} = \lim_{x \rightarrow 2} \frac{(x+2)(x-1)}{(x-1)(x+1)} = \frac{3}{2}$$

$$b) \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) = \infty - \infty = \lim_{x \rightarrow 2} \frac{x+2-4}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

$$c) \lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{4x}{x^2-9} \right) = \infty - \infty = \lim_{x \rightarrow 3} \frac{x+3-4x}{x^2-9} = \lim_{x \rightarrow 3} \frac{3-x}{x^2-9} = \lim_{x \rightarrow 3} \frac{-1}{x+3} = -\frac{1}{6}$$

$$d) \lim_{x \rightarrow +\infty} (\sqrt{x^2+1} - x) = \infty - \infty = \lim_{x \rightarrow +\infty} \frac{x^2+1-x^2}{(\sqrt{x^2+1}+x)} = \lim_{x \rightarrow +\infty} \frac{1}{(\sqrt{x^2+1}+x)} = 0$$

$$e) \lim_{x \rightarrow +\infty} (\sqrt{x^2-9} - \sqrt{x^2-6x}) = \infty - \infty = \lim_{x \rightarrow +\infty} \frac{x^2-9-x^2+6x}{(\sqrt{x^2-9}+\sqrt{x^2-6x})} = \lim_{x \rightarrow +\infty} \frac{6x-9}{(\sqrt{x^2-9}+\sqrt{x^2-6x})} = 3$$

$$f) \lim_{x \rightarrow +\infty} (\sqrt{x^2+x+1} - \sqrt{x^2-2x-1}) = \infty - \infty = \lim_{x \rightarrow +\infty} \frac{(x^2+x+1-x^2+2x+1)}{\sqrt{x^2+x+1}+\sqrt{x^2-2x-1}} =$$

$$= \lim_{x \rightarrow +\infty} \frac{3x+2}{\sqrt{x^2+x+1}+\sqrt{x^2-2x-1}} = \lim_{x \rightarrow +\infty} \frac{3x+2}{\sqrt{x^2+x+1}+\sqrt{x^2-2x-1}} = \lim_{x \rightarrow +\infty} \frac{3+\frac{2}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{2}{x}-\frac{1}{x^2}}} = \frac{3}{2}$$

10. Hallar a para que $\lim_{x \rightarrow 3} \frac{x^2-4x+3}{ax^2-18}$ sea finito.

Solución:

$$\lim_{x \rightarrow 3} \frac{x^2-4x+3}{ax^2-18} = \frac{0}{9a-18} \rightarrow 9a-18=0 \rightarrow a=2$$

$$\lim_{x \rightarrow 3} \frac{x^2-4x+3}{2x^2-18} = \lim_{x \rightarrow 3} \frac{(x-3)(x-1)}{2(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{x-1}{2(x+3)} = \frac{1}{6}$$

11. Calcular los siguientes límites:

$$a) \lim_{x \rightarrow +\infty} \left[x \cdot \log \left(1 + \frac{3}{x} \right) \right]$$

$$b) \lim_{x \rightarrow +\infty} \left(\frac{3x+1}{3x+4} \right)^{2x}$$

$$c) \lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2+4} \right)^{x^2}$$

$$d) \lim_{x \rightarrow 0} (ax+1)^{\frac{1}{x}}$$

$$e) \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{ax} \right)^x$$

$$f) \lim_{x \rightarrow 0} (\operatorname{sen} x + 1)^{\frac{1}{\operatorname{sen} x}}$$

$$\lim_{x \rightarrow +\infty} f(x)^{g(x)} = e^{\lim_{x \rightarrow +\infty} [f(x)-1] \cdot g(x)}$$

Solución:

$$a) \lim_{x \rightarrow +\infty} \left[x \cdot \log \left(1 + \frac{3}{x} \right) \right] = 0 \cdot \infty$$

$$\lim_{x \rightarrow +\infty} \left[\log \left(1 + \frac{3}{x} \right)^x \right] = \lim_{x \rightarrow +\infty} \left[\log \left(1 + \frac{1}{\frac{x}{3}} \right)^x \right] = \lim_{x \rightarrow +\infty} \left[\log \left(1 + \frac{1}{\frac{x}{3}} \right)^{\frac{3x}{3}} \right] = \log e^3 = 3$$

$$b) \lim_{x \rightarrow +\infty} \left(\frac{3x+1}{3x+4} \right)^{2x} = e^{-2}$$

$$\lim_{x \rightarrow +\infty} [f(x)-1] \cdot g(x) = \lim_{x \rightarrow +\infty} \left(\frac{3x+1}{3x+4} - 1 \right) \cdot 2x = \lim_{x \rightarrow +\infty} \left(\frac{-3}{3x+4} \right) \cdot 2x = \lim_{x \rightarrow +\infty} \left(\frac{-6x}{3x+4} \right) = -2$$

$$c) \lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2+4} \right)^{x^2} = e^{-3}$$

$$\lim_{x \rightarrow +\infty} [f(x)-1] \cdot g(x) = \lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2+4} - 1 \right) \cdot x^2 = \lim_{x \rightarrow +\infty} \left(\frac{-3}{x^2+4} \right) \cdot x^2 = \lim_{x \rightarrow +\infty} \left(\frac{-3x^2}{x^2+4} \right) = -3$$

$$d) \lim_{x \rightarrow 0} (ax+1)^{\frac{1}{x}} = e^a$$

$$\lim_{x \rightarrow 0} [f(x)-1] \cdot g(x) = \lim_{x \rightarrow 0} (ax+1-1) \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \left(\frac{ax}{x} \right) = a$$

$$e) \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{ax} \right)^x = e^{\frac{1}{a}}$$

$$\lim_{x \rightarrow +\infty} [f(x)-1] \cdot g(x) = \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{ax} - 1 \right) \cdot x = \lim_{x \rightarrow +\infty} \left(\frac{x}{ax} \right) = \frac{1}{a}$$

$$f) \lim_{x \rightarrow 0} (\operatorname{sen} x + 1)^{\frac{1}{\operatorname{sen} x}} = e$$

$$\lim_{x \rightarrow 0} [f(x)-1] \cdot g(x) = \lim_{x \rightarrow 0} (1 + \operatorname{sen} x - 1) \cdot \frac{1}{\operatorname{sen} x} = \lim_{x \rightarrow 0} \left(\frac{\operatorname{sen} x}{\operatorname{sen} x} \right) = 1$$