Problema 1 Calcular los siguientes límites:

a) $\lim\limits_{x\longrightarrow 0}\dfrac{x-\sin x}{\tan x-\sin x}$ (Castilla La Mancha 2005)

b) Si $f(x)=x^3e^{-x}$ calcular $\lim\limits_{x\longrightarrow\infty}f(x)$ y $\lim\limits_{x\longrightarrow-\infty}f(x)$ (Islas Baleares 2005)

c) $\lim\limits_{x\longrightarrow 3}\dfrac{\cos(\pi x)}{x}$ (La Rioja 2005)

d) $\lim\limits_{x\longrightarrow\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})$ (Madrid 2005)

e) $\lim\limits_{x\longrightarrow\infty}x\left[\arctan(e^x)-\dfrac{\pi}{2}\right]$ (Madrid 2005)

f) $\lim\limits_{x\longrightarrow 0}\dfrac{4x+\sin 2x}{\sin 3x}$ (Zaragoza)

g) $\lim\limits_{x\longrightarrow\infty}\left(\dfrac{2x-1}{2x}\right)^{x+2}$

h) $\lim\limits_{x\longrightarrow 0}x^x$

i) $\lim\limits_{x\longrightarrow\infty}\dfrac{\sqrt{3x^2+x-1}}{2x+3}$

j) $\lim\limits_{x\longrightarrow 0}\dfrac{\sqrt{3x+1}-\sqrt{x+1}}{x}$

Solución:

a) $\lim\limits_{x\longrightarrow 0}\dfrac{x-\sin x}{\tan x-\sin x}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\longrightarrow 0}\dfrac{1-\cos x}{1+\tan^2 x-\cos x}=\left[\dfrac{0}{0}\right]=$
$\lim\limits_{x\longrightarrow 0}\dfrac{\sin x}{2\tan x(1+\tan^2 x)+\sin x}=\lim\limits_{x\longrightarrow 0}\dfrac{\sin x}{2\tan x+2\tan^3 x+\sin x}=\left[\dfrac{0}{0}\right]=$
$\lim\limits_{x\longrightarrow 0}\dfrac{\cos x}{2(1+\tan^2 x)+6\tan^2 x(1+\tan^2 x)+\cos x}=\dfrac{1}{3}$

b) $\lim\limits_{x\longrightarrow\infty}f(x)=\lim\limits_{x\longrightarrow\infty}x^3e^{-x}=[\infty\cdot 0]=\lim\limits_{x\longrightarrow\infty}\dfrac{x^3}{e^x}=\left[\dfrac{\infty}{\infty}\right]=\lim\limits_{x\longrightarrow\infty}\dfrac{3x^2}{e^x}=$
$\left[\dfrac{\infty}{\infty}\right]=\lim\limits_{x\longrightarrow\infty}\dfrac{6x}{e^x}=\left[\dfrac{\infty}{\infty}\right]=\lim\limits_{x\longrightarrow\infty}\dfrac{6}{e^x}=0$

$$\lim_{x\longrightarrow-\infty}f(x)=\lim_{x\longrightarrow-\infty}x^3e^{-x}=\lim_{x\longrightarrow\infty}-x^3e^x=-\infty$$

c) $\lim\limits_{x \longrightarrow 3} \dfrac{\cos(\pi x)}{x} = -\dfrac{1}{3}$

d)

$$\lim_{x \longrightarrow \infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) = [\infty - \infty] =$$

$$\lim_{x \longrightarrow \infty} \frac{\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)\left(\sqrt{x^2+x} + \sqrt{x^2-x}\right)}{\sqrt{x^2+x} + \sqrt{x^2-x}} =$$

$$\lim_{x \longrightarrow \infty} \frac{\left(\sqrt{x^2+x}\right)^2 - \left(\sqrt{x^2-x}\right)^2}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x \longrightarrow \infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}} =$$

$$\lim_{x \longrightarrow \infty} \frac{\frac{2x}{x}}{\sqrt{\frac{x^2+x}{x^2}} + \sqrt{\frac{x^2-x}{x^2}}} = \frac{2}{2} = 1$$

e)

$$\lim_{x \longrightarrow \infty} x\left[\arctan\left(e^x\right) - \frac{\pi}{2}\right] = [0 \cdot \infty] = \lim_{x \longrightarrow \infty} \frac{\arctan\left(e^x\right) - \frac{\pi}{2}}{1/x} = \left[\frac{0}{0}\right] =$$

$$\lim_{x \longrightarrow \infty} \frac{\frac{e^x}{1+e^{2x}}}{-\frac{1}{x^2}} = \lim_{x \longrightarrow \infty} \frac{-x^2e^x}{1+e^{2x}} = \left[\frac{\infty}{\infty}\right] = \lim_{x \longrightarrow \infty} \frac{-2xe^x - x^2e^x}{2e^{2x}} =$$

$$\lim_{x \longrightarrow \infty} \frac{-2x - x^2}{2e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \longrightarrow \infty} \frac{-2 - 2x}{2e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \longrightarrow \infty} \frac{-2}{2e^x} = 0$$

f) $\lim\limits_{x \longrightarrow 0} \dfrac{4x + \sin 2x}{\sin 3x} = \left[\dfrac{0}{0}\right] = \lim\limits_{x \longrightarrow 0} \dfrac{4 + 2\cos 2x}{3\cos 3x} = \dfrac{6}{3} = 2$

g) $\lim\limits_{x \longrightarrow \infty} \left(\dfrac{2x-1}{2x}\right)^{x+2} = e^{-1/2}$

h) $\lim\limits_{x \longrightarrow 0} x^x = \lambda \Longrightarrow \lim\limits_{x \longrightarrow 0} x \ln x = \ln \lambda$

$\lim\limits_{x \longrightarrow 0} x \ln x = \lim\limits_{x \longrightarrow 0} \dfrac{\ln x}{x} = \left[\dfrac{-\infty}{\pm\infty}\right] = \lim\limits_{x \longrightarrow 0} \dfrac{1/x}{-1/x^2} = \lim\limits_{x \longrightarrow 0} (-x) = 0$

$\ln \lambda = 0 \Longrightarrow \lambda = 1$

i) $\lim\limits_{x \longrightarrow \infty} \dfrac{\sqrt{3x^2 + x - 1}}{2x + 3} = \dfrac{\sqrt{3}}{2}$

j) $\lim\limits_{x \longrightarrow 0} \dfrac{\sqrt{3x+1} - \sqrt{x+1}}{x} = 1$