Calcular las siguientes integrales:

1. $\displaystyle\int_{-2}^{-1} \frac{dx}{(x^2-x)(x-1)}$ (Andalucía Junio 2008)

   Solución:
   $$\int_{-2}^{-1} \frac{dx}{(x^2-x)(x-1)} = \int_{-2}^{-1} \frac{dx}{x(x-1)^2}$$
   Hacemos la descomposición en fracciones simples:
   $$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$
   $$1 = A(x-1)^2 + B(x(x-1) + Cx$$
   Si $x = 0 \Longrightarrow A = 1$
   Si $x = 1 \Longrightarrow C = 1$
   Si $x = 2 \Longrightarrow A + 2B + 2C = 1 \Longrightarrow C = -1$ luego:
   $$\int_{-2}^{-1} \frac{dx}{(x^2-x)(x-1)} = \int_{-2}^{-1} \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) =$$
   $$\left[\ln|x| - \ln|x-1| - \frac{1}{x-1}\right]_{-2}^{-1} = \frac{1}{6} + \ln\frac{3}{4}$$

2. $\displaystyle\int \frac{2x^3 - 9x^2 + 9x + 6}{x^2 - 5x + 6} dx$ (Castilla-La Mancha Junio 2008)

   Solución:
   $$\int \frac{2x^3 - 9x^2 + 9x + 6}{x^2 - 5x + 6} dx = \int \left(2x + 1 + \frac{2x}{(x-2)(x-3)}\right) dx$$
   Hacemos la descomposición en fracciones simples:
   $$\frac{2x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-2) + B(x-3)}{(x-2)(x-3)}$$
   $$2x = A(x-2) + B(x-3)$$
   Si $x = 2 \Longrightarrow A = -4$
   Si $x = 3 \Longrightarrow B = 6$
   $$\int \frac{2x^3 - 9x^2 + 9x + 6}{x^2 - 5x + 6} dx = \int \left(2x + 1 - \frac{4}{x-2} + \frac{6}{x-3}\right) dx =$$
   $$= x^2 + x - 4\ln|x-2| + 6\ln|x-3| + C$$

3. $\displaystyle\int_0^{\pi} e^x \sin x \, dx$ (Castilla-La Mancha Junio 2008)

Solución:

Se trata de una integral por partes, hacemos

$$\begin{array}{l} u = \sin x \Longrightarrow du = \cos x \, dx \\ dv = e^x \, dx \Longrightarrow v = e^x \end{array}$$

$$A = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Volvemos a integrar por partes:

$$\begin{array}{l} u = \cos x \Longrightarrow du = -\sin x \, dx \\ dv = e^x \, dx \Longrightarrow v = e^x \end{array}$$

$$A = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx =$$

$$e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Es decir:

$$A = e^x \sin x - e^x \cos x - A \Longrightarrow A = \frac{1}{2} e^x (\sin x - \cos x)$$

$$\int_0^{\pi} e^x \sin x \, dx = \left[\frac{1}{2} e^x (\sin x - \cos x) \right]_0^{\pi} = \frac{1}{2}(e^{\pi} + 1)$$

Calcular los siguientes límites:

1. $\displaystyle\lim_{x \longrightarrow 0} \frac{x^3 - 8x^2 + 7x}{x^2 - x}$ (Castilla-La Mancha Junio 2008)

   Solución: $\displaystyle\lim_{x \longrightarrow 0} \frac{x(x^2 - 8x + 7)}{x(x-1)} = -7$

2. $\displaystyle\lim_{x \longrightarrow \frac{\pi}{2}} \left(\frac{2x}{\pi} + \cos x \right)^{1/\cos x}$ Castilla-La Mancha (Junio 2008)

   Solución:

$$\ln \lambda = \lim_{x \longrightarrow \frac{\pi}{2}} \ln \left(\left(\frac{2x}{\pi} + \cos x \right)^{1/\cos x} \right) = \lim_{x \longrightarrow \frac{\pi}{2}} \frac{\ln \left(\frac{2x}{\pi} + \cos x \right)}{\cos x} =$$

$$\lim_{x \longrightarrow \frac{\pi}{2}} \frac{\frac{2/\pi - \sin x}{2x/\pi + \cos x}}{-\sin x} = \lim_{x \longrightarrow \frac{\pi}{2}} \frac{\frac{2}{\pi} - \sin x}{-\sin x \left(\frac{2x}{\pi} + \cos x \right)} = 1 - \frac{2}{\pi}$$

$$\ln \lambda = 1 - \frac{2}{\pi} \Longrightarrow \lambda = e^{1 - \frac{2}{\pi}}$$