

INTEGRACIÓN DE FUNCIONES RACIONALES

Mediante el recurso de la descomposición en fracciones simples, el proceso de integración de funciones racionales se puede simplificar notablemente.

EJERCICIOS DESARROLLADOS

7.1.-Encontrar: $\int \frac{dx}{x^2-9}$

Solución.- Descomponiendo el denominador en factores: $x^2 - 9 = (x+3)(x-3)$,
Como los factores son ambos lineales y diferentes se tiene:

$$\frac{1}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}, \text{ de donde:}$$

$$\frac{1}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3} \Rightarrow 1 = A(x-3) + B(x+3) (*) \Rightarrow 1 = (A+B)x + (-3A+3B)$$

Para calcular las constantes A y B, se pueden identificar los coeficientes de igual potencia x en la última expresión, y se resuelve el sistema de ecuaciones dado; obteniendo así los valores de las constantes en referencia (método general) luego:

$$\begin{pmatrix} A+B=0 \\ -3A+3B=1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3A+3B=0 \\ -3A+3B=1 \end{pmatrix} \Rightarrow 6B=1 \Rightarrow B = \frac{1}{6}, \text{ además:}$$

$$A+B=0 \Rightarrow A=-B \Rightarrow A = -\frac{1}{6}$$

También es frecuente usar otro mecanismo, que consiste en la expresión (*)

Sustituyendo a x por los valores que anulen los denominadores de las fracciones:

$$x=3 \Rightarrow 1=6B \Rightarrow B = \frac{1}{6}$$

$$x=-3 \Rightarrow 1=-6A \Rightarrow A = -\frac{1}{6}$$

Usando cualquier método de los señalados anteriormente, se establece que:

$$\frac{1}{x^2-9} = \frac{-\frac{1}{6}}{x+3} + \frac{\frac{1}{6}}{x-3}, \text{ Luego se tiene:}$$

$$\begin{aligned} \int \frac{dx}{x^2-9} &= -\frac{1}{6} \int \frac{dx}{x+3} + \frac{1}{6} \int \frac{dx}{x-3} = -\frac{1}{6} \ell \eta |x+3| + \frac{1}{6} \ell \eta |x-3| + c \\ &= \frac{1}{6} (\ell \eta |x-3| - \ell \eta |x+3|) + c \end{aligned}$$

Respuesta: $\int \frac{dx}{x^2-9} = \frac{1}{6} \ell \eta \left| \frac{x-3}{x+3} \right| + c$

7.2.-Encontrar: $\int \frac{dx}{x^2+7x-6}$

Solución.- Sea: $x^2+7x+6 = (x+6)(x+1)$, factores lineales y diferentes; luego:

$$\frac{1}{x^2+7x+6} = \frac{A}{x+6} + \frac{B}{x+1},$$

De donde:

$1 = A(x+1) + B(x+6)$ (*) $\Rightarrow 1 = (A+B)x + (A+6B)$, calculando las constantes A y B por el método general, se tiene: $1 = (A+B)x + (A+6B)$

$$\begin{pmatrix} A+B=0 \\ A+6B=1 \end{pmatrix} \Rightarrow -\begin{pmatrix} -A-B=0 \\ A+6B=1 \end{pmatrix} \Rightarrow 5B=1 \Rightarrow B = \frac{1}{5}, \text{ además:}$$

$$A+B=0 \Rightarrow A = -B \Rightarrow A = -\frac{1}{5}$$

Ahora utilizando el método abreviado se tiene:

$$x = -1 \Rightarrow 1 = 5B \Rightarrow B = \frac{1}{5}$$

$$x = -6 \Rightarrow 1 = -5A \Rightarrow A = -\frac{1}{5}$$

Usando cualquier método se puede establecer:

$$\frac{1}{x^2+7x+6} = \frac{-\frac{1}{5}}{x+6} + \frac{\frac{1}{5}}{x+1}, \text{ Luego se tiene:}$$

$$\begin{aligned} \int \frac{dx}{x^2+7x+6} &= -\frac{1}{5} \int \frac{dx}{x+6} + \frac{1}{5} \int \frac{dx}{x+1} = -\frac{1}{5} \ell \eta |x+6| + \frac{1}{5} \ell \eta |x+1| + c \\ &= \frac{1}{5} (\ell \eta |x+1| - \ell \eta |x+6|) + c \end{aligned}$$

Respuesta: $\int \frac{dx}{x^2+7x+6} = \frac{1}{5} \ell \eta \left| \frac{x+1}{x+6} \right| + c$

7.3.-Encontrar: $\int \frac{xdx}{x^2-4x+4}$

Solución.- Sea: $x^2-4x+4 = (x-2)^2$, factores lineales con repetición; luego:

$$\frac{x}{x^2-x+4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \Rightarrow \frac{x}{\cancel{x^2-x+4}} = \frac{A(x-2)+B}{(x-2)^2},$$

De donde:

$x = A(x-2) + B$ (*), calculando las constantes A y B por el método general, se tiene: $x = Ax + (-2A + B)$, luego:

$$\begin{pmatrix} A = 1 \\ -2A + B = 0 \end{pmatrix} \Rightarrow B = 2A \Rightarrow B = 2(1) \Rightarrow B = 2$$

Usando el método abreviado, se sustituye en x , el valor que anula el denominador (o los denominadores), y si este no es suficiente se usan para sustituir cualquier valor conveniente de x , esto es: $x=0, x=-1$; luego en (*)

$$x=2 \Rightarrow 2=B \Rightarrow B=2$$

$$x=0 \Rightarrow 0=-2A+B \Rightarrow 2A+B \Rightarrow A=\frac{B}{2} \Rightarrow A=1$$

Usando cualquier método se establece:

$$\int \frac{x dx}{x^2-4x+4} = \int \frac{dx}{x-2} + 2 \int \frac{dx}{(x-2)^2} = \ell \eta |x-2| - \frac{2}{x-2} + c$$

Respuesta: $\int \frac{x dx}{x^2-4x+4} = \ell \eta |x-2| - \frac{2}{x-2} + c$

7.4.-Encontrar: $\int \frac{(2x^2+3)dx}{x^3-2x^2+x}$

Solución.- Sea: $x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$, factores lineales:

$x, x-1$; donde este último es con repetición; luego:

$$\frac{2x^2+3}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \Rightarrow \frac{2x^2+3}{\cancel{x^3-2x^2+x}} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{\cancel{x(x-1)^2}}$$

De donde:

$2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$ (*), calculando las constantes A y B por el método general, se tiene: $2x^2+3 = (A+B)x^2 + (-2A-B+C)x + A$, de donde identificando los coeficientes de igual potencia de x se puede obtener el siguiente sistema de ecuaciones:

$$\begin{pmatrix} A+B & = & 2 \\ -2A-B+C & = & 0 \\ A & = & 3 \end{pmatrix} \Rightarrow B=2-A \Rightarrow B=2-3 \Rightarrow B=-1, \text{ tomando la segunda ecuación}$$

del sistema: $C=2A+B \Rightarrow C=2(3)-1 \Rightarrow C=5$, también es posible usar el método abreviado, utilizando para ello la expresión (*) en la cual:

$$x=1 \Rightarrow 2(1)+3=C \Rightarrow C=5$$

$$x=0 \Rightarrow 3=A \Rightarrow A=3$$

Usando un valor arbitrario para x , sea este $x=-1$:

$$x=-1 \Rightarrow 2(-1)^2+3 = A(-2)^2 + B(-1)(-2) + C(-1) \Rightarrow 5 = 4A + 2B - C, \text{ luego:}$$

$$2B = 5 - 4A + C \Rightarrow 2B = 5 - 4(3) + 5 \Rightarrow 2B = -2 \Rightarrow B = -1, \text{ S, e establece que:}$$

$$\frac{2x^2+3}{x^3-2x^2+x} = \frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}, \text{ entonces:}$$

$$\frac{2x^2+3}{x^3-2x^2+x} = 3 \int \frac{dx}{x} - \int \frac{dx}{x-1} + 5 \int \frac{dx}{(x-1)^2} = 3\ell \eta |x| - \ell \eta |x-1| - \frac{5}{x-1} + c$$

Respuesta: $\int \frac{(2x^2+3)dx}{x^3-2x^2+x} = \ell \eta \left| \frac{x^3}{x-1} \right| - \frac{5}{x-1} + c$

7.5.-Encontrar: $\int \frac{dx}{x^3 - 2x^2 + x}$

Solución.- $x^3 - 2x^2 + x = x(x-1)^2$, factores lineales:

$x, x-1$; donde este último es con repetición; luego:

$$\frac{1}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \Rightarrow \frac{1}{\cancel{x^3 - 2x^2 + x}} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

De donde:

$1 = A(x-1)^2 + Bx(x-1) + Cx(*)$, calculando las constantes A y B por el método general, se tiene: $1 = (A+B)x^2 + (-2A-B+C)x + A$, de donde identificando los coeficientes de igual potencia de x se puede obtener el siguiente sistema de ecuaciones:

$$\begin{pmatrix} A+B & = & 0 \\ -2A-B+C & = & 0 \\ A & = & 1 \end{pmatrix} \Rightarrow B = -A \Rightarrow B = -1, \text{ tomando la segunda ecuación del}$$

sistema: $C = 2A + B \Rightarrow C = 2(1) - 1 \Rightarrow C = 1$, a partir de lo cual se tiene:

$$\frac{1}{x^3 - 2x^2 + x} = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

$$\int \frac{dx}{x^3 - 2x^2 + x} = \int \frac{dx}{x} - \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \ell \eta |x| - \ell \eta |x-1| - \frac{1}{x-1} + c$$

Respuesta: $\int \frac{dx}{x^3 - 2x^2 + x} = \ell \eta \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + c$

7.6.-Encontrar: $\int \frac{x^4 - 6x^3 + 12x^2 + 6}{x^3 - 6x^2 + 12x - 8} dx$

Solución.- Se sabe que si el grado del polinomio dividendo, es igual o superior al grado del polinomio divisor, previamente conviene efectuar la división de tales polinomios.

$$\begin{array}{r} x^4 - 6x^3 + 12x^2 + 0x + 6 \quad | \quad x^3 - 6x^2 + 12x - 8 \\ \underline{-x^4 + 6x^3 - 12x^2 + 8x} \qquad \qquad \qquad x \\ \quad \quad \quad 8x + 6 \end{array}$$

Luego se tiene: $\int \frac{x^4 - 6x^3 + 12x^2 + 6}{x^3 - 6x^2 + 12x - 8} dx = \int x dx + \int \frac{(8x+6)dx}{x^3 - 6x^2 + 12x - 8}$

La descomposición de: $x^3 - 6x^2 + 12x - 8$:

$$\begin{array}{r|rrrr} 1 & -6 & 12 & -8 \\ 2 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array} \quad x = 2 \Rightarrow (x-2)$$

$$x^2 - 4x + 4 = (x-2)^2$$

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3$$

Esto es factores lineales: $[(x-2)]$ con repetición por tanto:

$$\frac{8x+6}{x^3-6x^2+12x-8} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

$$\frac{8x+6}{\cancel{x^3-6x^2+12x-8}} = \frac{A(x-2)^2 + B((x-2) + C}{(x-2)^3}$$

Luego:

$$8x+6 = A(x-2)^2 + B(x-2) + C \Rightarrow 8x+6 = A(x^2-4x+4) + B(x-2) + C$$

$$8x+6 = Ax^2 + (-4A+B)x + (4A-2B+C)$$

Calculando las constantes A y B por el método general, se tiene:

$$\begin{cases} A & = 0 \\ -4A + B & = 8 \\ +4A - 2B + C & = 6 \end{cases} \Rightarrow B = 8 + 4A \Rightarrow B = 8 + 4(0) \Rightarrow B = 8,$$

Resolviendo el sistema: $C = 6 - 4A + 2B \Rightarrow C = 6 - 4(0) + 2(8) \Rightarrow C = 22$, luego:

$$\frac{8x+6}{x^3-6x^2+12x-8} = \frac{0}{x-2} + \frac{8}{(x-2)^2} + \frac{22}{(x-2)^3}, \text{ de donde:}$$

$$\int \frac{(8x+6)dx}{x^3-6x^2+12x-8} = 8 \int \frac{dx}{(x-2)^2} + 22 \int \frac{dx}{(x-2)^3}, \text{ o sea:}$$

$$= \int x dx + 8 \int \frac{dx}{(x-2)^2} + 22 \int \frac{dx}{(x-2)^3} = \int x dx + 8 \int (x-2)^{-2} dx + 22 \int (x-2)^{-3} dx$$

$$\frac{x^2}{2} - \frac{8}{x-2} - \frac{11}{(x-2)^2} + c$$

Respuesta: $\int \frac{x^4-6x^3+12x^2+6}{x^3-6x^2+12x-8} dx = \frac{x^2}{2} - \frac{8}{x-2} - \frac{11}{(x-2)^2} + c$

7.7.-Encontrar: $\int \frac{x^3+x^2+x+3}{x^4+4x^2+3} dx$

Solución.- $x^4+4x^2+3 = (x^2+3)(x^2+1)$, la descomposición es en factores cuadráticos sin repetición, por lo tanto:

$$\frac{x^3+x^2+x+3}{x^4+4x^2+3} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^3+x^2+x+3}{\cancel{x^4+4x^2+3}} = \frac{(Ax+B)(x^2+1) + (Cx+D)(x^2+3)}{(x^2+3)(x^2+1)}$$

$$x^3+x^2+x+3 = A(x^3+x) + B(x^2+1) + C(x^3+3x) + D(x^2+3)$$

$$x^3+x^2+x+3 = (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D), \text{ luego:}$$

$$\begin{cases} (1) & A + C = 1 \\ (2) & B + D = 1 \\ (3) & A + 3C = 1 \\ (4) & B + 3D = 3 \end{cases}$$

Con (1) y (3), se tiene: $\begin{cases} A + C = 1 \\ A + 3C = 1 \end{cases} \Rightarrow A = 1, C = 0$

Con (2) y (4), se tiene: $\begin{cases} B + D = 1 \\ B + 3D = 3 \end{cases} \Rightarrow B = 0, D = 1$

Por lo tanto: $\frac{x^3 + x^2 + x + 3}{x^4 + 4x^2 + 3} = \frac{x}{x+3} + \frac{1}{x^2+1}$, o sea:

$$\int \frac{x^3 + x^2 + x + 3}{x^4 + 4x^2 + 3} dx = \int \frac{xdx}{x+3} + \int \frac{dx}{x^2+1}, \text{ sea: } u = x^2 + 3, du = 2xdx, \text{ luego:}$$

$$\begin{aligned} \int \frac{x^3 + x^2 + x + 3}{x^4 + 4x^2 + 3} dx &= \frac{1}{2} \int \frac{2xdx}{x+3} + \int \frac{dx}{x^2+1} = \frac{1}{2} \int \frac{du}{u} + \int \frac{dx}{x^2+1} \\ &= \frac{1}{2} \ell \eta |u| + \text{arc } \tau gx + c = \frac{1}{2} \ell \eta |x^2 + 3| + \text{arc } \tau gx + c \end{aligned}$$

Respuesta: $\int \frac{x^3 + x^2 + x + 3}{x^4 + 4x^2 + 3} dx = \frac{1}{2} \ell \eta |x^2 + 3| + \text{arc } \tau gx + c$

7.8.-Encontrar: $\int \frac{x^4 dx}{x^4 + 2x^2 + 1}$

Solución.-

$$\frac{x^4}{x^4 + 2x^2 + 1} = \frac{-x^4 - 2x^2 - 1}{x^4 + 2x^2 + 1} + \frac{1}{x^4 + 2x^2 + 1}$$

Luego $\int \frac{x^4 dx}{x^4 + 2x^2 + 1} = \int \left(1 - \frac{2x^2 + 1}{x^4 + 2x^2 + 1} \right) dx = \int dx - \int \frac{2x^2 + 1}{x^4 + 2x^2 + 1} dx$

La descomposición del denominador es: $x^4 + 2x^2 + 1 = (x^2 + 1)^2$, entonces:

$$\frac{2x^2 + 1}{x^4 + 2x^2 + 1} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \Rightarrow \frac{2x^2 + 1}{x^4 + 2x^2 + 1} = \frac{(Ax + B)(x^2 + 1)(Cx + D)}{(x^2 + 1)^2}$$

$$2x^2 + 1 = (Ax + B)(x^2 + 1) + (Cx + D) \Rightarrow 2x^2 + 1 = A(x^3 + x) + B(x^2 + 1) + Cx + D$$

$$2x^2 + 1 = Ax^3 + Bx^2 + (A + C)x + (B + D)$$

Calculando las constantes por el método general, se tiene:

$$\begin{cases} A & = 0 \\ B & = 2 \\ A + C & = 0 \\ B + D & = 1 \end{cases}$$

Resolviendo el sistema: $C = -A \Rightarrow A = 0 \therefore C = 0$, $B + D = 1 \Rightarrow D = 1 - B \Rightarrow D = -1$
 luego:

$$\frac{2x^2 + 1}{x^4 + 2x^2 + 1} = \frac{2}{x^2 + 1} - \frac{1}{(x^2 + 1)^2}, \text{ o sea:}$$

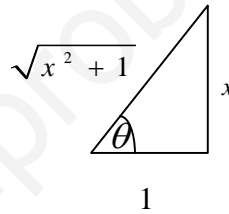
$$\int \frac{2x^2 + 1}{x^4 + 2x^2 + 1} = 2 \int \frac{dx}{x^2 + 1} - \int \frac{dx}{(x^2 + 1)^2} = 2 \int \frac{dx}{x^2 + 1} - \int \frac{dx}{(\sqrt{x^2 + 1})^4}$$

Sea: $x = \tau g \theta$, $dx = \sec^2 \theta d\theta$; $\sqrt{x^2 + 1} = \sec \theta$, luego:

$$= 2 \operatorname{arc} \tau g x - \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = 2 \operatorname{arc} \tau g x - \int \frac{d\theta}{\sec^2 \theta} = 2 \operatorname{arc} \tau g x - \int \cos^2 \theta$$

$$= 2 \operatorname{arc} \tau g x - \int \frac{1 + \cos 2\theta}{2} d\theta = 2 \operatorname{arc} \tau g x - \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta$$

$$\operatorname{arc} \tau g x - \frac{1}{2} \theta - \frac{1}{2} \operatorname{sen} 2\theta + c = 2 \operatorname{arc} \tau g x - \frac{1}{2} \theta - \frac{1}{2} \operatorname{sen} \theta \cos \theta + c$$



De la figura se tiene que:

$$\tau g \theta = x, \theta \operatorname{arc} \tau g \theta, \operatorname{sen} \theta = \frac{x}{\sqrt{x^2 + 1}}, \cos \theta = \frac{1}{\sqrt{x^2 + 1}}$$

$$\text{Luego: } = 2 \operatorname{arc} \tau g x - \frac{1}{2} \operatorname{arc} \tau g x - \frac{1}{2} \frac{x}{\sqrt{x^2 + 1}} \frac{1}{\sqrt{x^2 + 1}} + c = 2 \operatorname{arc} \tau g x - \frac{1}{2} \operatorname{arc} \tau g x - \frac{x}{2(x^2 + 1)} + c$$

Recordando que:

$$\frac{x^4 dx}{x^4 + 2x^2 + 1} = \int dx - \int \frac{(2x^2 + 1) dx}{x^4 + 2x^2 + 1} = x - 2 \operatorname{arc} \tau g x + \frac{1}{2} \operatorname{arc} \tau g x + \frac{1}{2} \frac{x}{(x^2 + 1)} + c$$

$$\text{Respuesta: } \int \frac{x^4 dx}{x^4 + 2x^2 + 1} = x - \frac{3}{2} \operatorname{arc} \tau g x + \frac{x}{2(x^2 + 1)} + c$$

7.9.-Encontrar: $\int \frac{x^4 dx}{x^4 - 1}$

Solución.-

$$\frac{x^4}{-x^4 + 1} \quad \frac{|x^4 - 1|}{1}$$

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Luego:

$$\int \frac{x^4 dx}{x^4 - 1} = \int \left(1 + \frac{1}{x^4 - 1} \right) dx = \int dx + \int \frac{dx}{x^4 - 1}$$

Descomponiendo en factores el denominador:

$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x^2 + 1)(x + 1)(x - 1)$, es decir factores lineales y cuadráticos sin repetición por tanto:

$$\frac{1}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$\frac{1}{x^4-1} = \frac{(Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x+1)(x^2+1)}{(x^2+1)(x+1)(x-1)}$$

$$1 = A(x^3-x) + B(x^2+1) + C(x^3-x^2+x-1) + D(x^3+x^2+x+1)$$

$$1 = (A+C+D)x^3 + (B-C+D)x^2 + (-A+C+D)x + (-B-C+D)$$

Luego:

$$\begin{cases} (1) & A + C + D = 0 \\ (2) & B - C + D = 0 \\ (3) & -A + C + D = 0 \\ (4) & -B - C + D = 1 \end{cases}$$

Con (1) y (3), se tiene: $\begin{cases} A + C + D = 0 \\ -A + C + D = 0 \end{cases} \Rightarrow 2C + 2D = 0$ **(5)**

Con (2) y (4), se tiene: $\begin{cases} B - C + D = 0 \\ -B - C + D = 1 \end{cases} \Rightarrow -2C + 2D = 1$ **(6)**

Con (5) y (6), se tiene: $\begin{cases} 2C + 2D = 0 \\ -2C + 2D = 1 \end{cases} \Rightarrow C = -\frac{1}{4}, D = \frac{1}{4}$

Además: $A = 0, B = -\frac{1}{2}$, luego:

$$\frac{1}{x^4-1} = -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}, \text{ con lo cual:}$$

$$\int \frac{dx}{x^4-1} = -\frac{1}{2} \int \frac{dx}{(x^2+1)} - \frac{1}{4} \int \frac{dx}{(x+1)} + \frac{1}{4} \int \frac{dx}{(x-1)}$$

$$= -\frac{1}{2} \arctan x - \frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| + c$$

Dado que: $\int \frac{x^4 dx}{x^4-1} = \int dx + \int \frac{dx}{x^4-1} = x - \frac{1}{2} \arctan x + \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + c$, entonces:

Respuesta: $\int \frac{1}{x^4-1} = x - \frac{1}{2} \arctan x + \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + c$

7.10.-Encontrar: $\int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x} dx$

Solución.-

$$x^4 - 2x^3 + 3x^2 - x + 3 \quad | \quad x^3 - 2x^2 + 3x$$

$$-x^4 + 2x^3 - 3x^2 \quad \quad \quad x$$

$$\boxed{-x+3}$$

Luego:

$$\int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x} dx = \int \left(x - \frac{x-3}{x^3 - 2x^2 + 3x} \right) dx = \int x dx - \int \frac{x-3}{x^3 - 2x^2 + 3x} dx$$

Descomponiendo en factores el denominador:

$x^3 - 2x^2 + 3x = x(x^2 - 2x + 3)$, es decir un factor lineal y uno cuadrático; por lo cual:

$$\frac{x-3}{x^3 - 2x^2 + 3x} = \frac{A}{x} + \frac{Bx+C}{x^2 - 2x + 3} \Rightarrow \frac{x-3}{x^3 - 2x^2 + 3x} = \frac{A(x^2 - 2x + 3) + (Bx + C)x}{x(x^2 - 2x + 3)}$$

$$x-3 = A(x^2 - 2x + 3) + (Bx + C)x \Rightarrow x-3 = (A+B)x^2 + (-2A+C)x + 3A$$

De donde:

$$\begin{cases} A+B = 0 \\ -2A + C = 1 \\ 3A = -3 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = -A \Rightarrow B = 1 \\ C = 1 + 2A \Rightarrow C = -1 \end{cases}$$

Luego:

$$\frac{x-3}{x^3 - 2x^2 + 3x} = -\frac{1}{x} + \frac{x-1}{x^2 - 2x + 3}, \text{ de donde:}$$

$$\int \frac{x-3}{x^3 - 2x^2 + 3x} dx = -\int \frac{dx}{x} + \int \frac{x-1}{x^2 - 2x + 3} dx = -\ell \eta |x| + \int \frac{x-1}{x^2 - 2x + 3} dx$$

$$\begin{aligned} \int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x} dx &= \int x dx + \ell \eta |x| - \int \frac{x-1}{x^2 - 2x + 3} dx \\ &= \frac{x^2}{2} + \ell \eta |x| - \int \frac{x-1}{x^2 - 2x + 3} dx = \frac{x^2}{2} + \ell \eta |x| - \frac{1}{2} \int \frac{2(x-1)dx}{x^2 - 2x + 3} \end{aligned}$$

$$\text{Sea: } u = x^2 - 2x + 3, du = (2x - 2)dx \Rightarrow du = 2(x - 1)dx$$

$$= \frac{x^2}{2} + \ell \eta |x| - \frac{1}{2} \int \frac{du}{u} = \frac{x^2}{2} + \ell \eta |x| - \frac{1}{2} \ell \eta |x^2 - 2x + 3| + c$$

$$\text{Respuesta: } \int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x} dx = \frac{x^2}{2} + \ell \eta \left| \frac{x}{\sqrt{x^2 - 2x + 3}} \right| + c$$

EJERCICIOS PROPUESTOS

Usando La técnica de descomposición en fracciones simples parciales, calcular las siguientes integrales:

$$7.11.- \int \frac{(x^5 + 2)dx}{x^2 - 1}$$

$$7.12.- \int \frac{xdx}{(x+1)^2}$$

$$7.13.- \int \frac{x^3 dx}{x^2 - 2x - 3}$$

$$7.14.- \int \frac{(3x+7)dx}{(x-1)(x-2)(x-3)}$$

$$7.15.- \int \frac{dx}{x^3 + 1} dx$$

$$7.16.- \int \frac{(x+5)dx}{x^2 - x + 6}$$

$$7.17.- \int \frac{(x^2 + 1)dx}{x^3 + 1}$$

$$7.18.- \int \frac{(x^2 + 6)dx}{(x-1)^2(x-2)}$$

$$7.19.- \int \frac{(x^2 - 1)dx}{(x^2 + 1)(x-2)}$$

$$\begin{array}{lll}
7.20.- \int \frac{x dx}{x^2 - 4x - 5} & 7.21.- \int \frac{x dx}{x^2 - 2x - 3} & 7.22.- \int \frac{(x+1) dx}{x^2 + 4x - 5} \\
7.23.- \int \frac{x^2 dx}{x^2 + 2x + 1} & 7.24.- \int \frac{dx}{x(x+1)^2} & 7.25.- \int \frac{dx}{(x+1)(x^2 + 1)} \\
7.26.- \int \frac{dx}{x(x^2 + x + 1)} & 7.27.- \int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx & 7.28.- \int \frac{(x^2 + 2x + 3) dx}{(x-1)(x+1)^2} \\
7.29.- \int \frac{3x^2 + 2x - 2}{x^3 - 1} dx & 7.30.- \int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2 + 2)^2} dx & 7.31.- \int \frac{(2x^2 - 7x - 1) dx}{x^3 + x^2 - x - 1} \\
7.32.- \int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} dx & 7.33.- \int \frac{x^3 + 7x^2 - 5x + 5}{(x-1)^2(x+1)^2} dx & 7.34.- \int \frac{2x dx}{(x^2 + x + 1)^2} \\
7.35.- \int \frac{x^2 + 2x + 3}{x^3 - x} dx & 7.36.- \int \frac{(2x^2 - 3x + 5) dx}{(x+2)(x-1)(x-3)} & 7.37.- \int \frac{(3x^2 + x - 2) dx}{(x-1)(x^2 + 1)} \\
7.38.- \int \frac{(x+5) dx}{x^3 - 3x + 2} & 7.39.- \int \frac{2x^3 + 3x^2 + x - 1}{(x+1)(x^2 + 2x + 2)^2} dx & 7.40.- \int \frac{(2x+1) dx}{3x^3 + 2x - 1} \\
7.41.- \int \frac{(2x^2 + 3x - 1) dx}{x^3 + 2x^2 + 4x + 2} & 7.42.- \int \frac{x^4 - 2x^2 + 3x + 4}{(x-1)^3(x^2 + 2x + 2)} dx & 7.43.- \int \frac{e^t dt}{e^{2t} + 3e^t + 2} \\
7.44.- \int \frac{\operatorname{sen} \theta d\theta}{\cos^2 \theta + \cos \theta - 2} & 7.45.- \int \frac{4x^4 - 2x^3 - x^2 + 3x + 1}{(x^3 + x^2 - x - 1)} dx & 7.46.- \int \frac{3x^4 dx}{(x^2 + 1)^2} \\
7.47.- \int \frac{(2x^2 + 41x - 91) dx}{x^3 - 2x^2 - 11x + 12} & 7.48.- \int \frac{(2x^4 + 3x^3 - x - 1) dx}{(x-1)(x^2 + 2x + 2)^2} & 7.49.- \int \frac{dx}{e^{2x} + e^x - 2} \\
7.50.- \int \frac{\operatorname{sen} x dx}{\cos x(1 + \cos^2 x)} & 7.51.- \int \frac{(2 + \tau g^2 \theta) \sec^2 \theta d\theta}{1 + \tau g^3 \theta} & 7.52.- \int \frac{(5x^3 + 2) dx}{x^3 - 5x^2 + 4x} \\
7.53.- \int \frac{x^5 dx}{(x^3 + 1)(x^3 + 8)} & &
\end{array}$$

RESPUESTAS

$$7.11.- \int \frac{(x^5 + 2) dx}{x^2 - 1}$$

Solución.-

$$\int \frac{(x^5 + 2) dx}{x^2 - 1} = \int \left(x^3 + x + \frac{x+2}{x^2 - 1} \right) dx = \int x^3 dx + \int x dx + \int \frac{x+2}{x^2 - 1} dx$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + \int \frac{(x+2) dx}{(x+1)(x-1)} (*) , \text{ luego:}$$

$$\frac{x+2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow x+2 = A(x-1) + B(x+1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 3=2B \Rightarrow B=3/2 \\ x=-1 \Rightarrow 1=-2A \Rightarrow A=-1/2 \end{cases}$$

$$\begin{aligned} (*) &= \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{2} \int \frac{dx}{x+1} + \frac{3}{2} \int \frac{dx}{x-1} = \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{2} \ell \eta |x+1| + \frac{3}{2} \ell \eta |x-1| + c \\ &= \frac{x^4}{4} + \frac{x^2}{2} + \eta \left| \frac{(x-1)^{3/2}}{\sqrt{x+1}} \right| + c \end{aligned}$$

$$7.12.- \int \frac{xdx}{(x+1)^2}$$

Solución.-

$$\int \frac{xdx}{(x+1)^2} = \int \frac{Adx}{x+1} + \int \frac{Bdx}{(x+1)^2} \quad (*), \text{ luego:}$$

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x = A(x+1) + B$$

$$\therefore \begin{cases} x=-1 \Rightarrow -1=B \\ x=0 \Rightarrow 0=A+B \Rightarrow A=-B \Rightarrow A=-1 \end{cases}$$

$$(*) \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \ell \eta |x+1| + (x+1)^{-1} + c = \ell \eta |x+1| + \frac{1}{x+1} + c$$

$$7.13.- \int \frac{x^3 dx}{x^2 - 2x - 3}$$

Solución.-

$$\int \frac{x^3 dx}{x^2 - 2x - 3} = \int \left(x+2 + \frac{7x+6}{x^2 - 2x - 3} \right) dx = \int x dx + 2 \int dx + \int \frac{(7x+6)dx}{x^2 - 2x - 3}$$

$$= \frac{x^2}{2} + 2x + \int \frac{(7x+6)dx}{(x-3)(x+1)} \quad (*), \text{ luego:}$$

$$\frac{(7x+6)}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \Rightarrow 7x+6 = A(x+1) + B(x-3)$$

$$\therefore \begin{cases} x=3 \Rightarrow 27=4A \Rightarrow A=27/4 \\ x=-1 \Rightarrow -1=-4B \Rightarrow B=1/4 \end{cases}$$

$$(*) = \frac{x^2}{2} + 2x + \frac{27}{4} \int \frac{dx}{x-3} + \frac{1}{4} \int \frac{dx}{x+1} = \frac{x^2}{2} + 2x + \frac{27}{4} \ell \eta |x-3| + \frac{1}{4} \ell \eta |x+1| + c$$

$$= \frac{x^2}{2} + 2x + \frac{1}{4} \ell \eta |(x-3)^{27}(x+1)| + c$$

$$7.14.- \int \frac{(3x+7)dx}{(x-1)(x-2)(x-3)}$$

Solución.-

$$\int \frac{(3x+7)dx}{(x-1)(x-2)(x-3)} = \int \frac{Adx}{x-1} + \int \frac{Bdx}{x-2} + \int \frac{Cdx}{x-3} \quad (*)$$

$$\frac{(3x+7)}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$3x-7 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$, luego:

$$\therefore \begin{cases} x=1 \Rightarrow -4 = 2A \Rightarrow A = -2 \\ x=2 \Rightarrow -1 = -B \Rightarrow B = 1 \\ x=3 \Rightarrow 2 = 2C \Rightarrow C = 1 \end{cases}$$

$$\begin{aligned} (*) &= -2 \int \frac{dx}{x-1} + \int \frac{dx}{x-2} + \int \frac{dx}{x-3} = -2\ell\eta|x-1| + \ell\eta|x-2| + \ell\eta|x-3| + c \\ &= \ell\eta \left| \frac{(x-2)(x-3)}{(x-1)^2} \right| + c \end{aligned}$$

7.15.- $\int \frac{dx}{x^3+1} dx$

Solución.-

$$\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} = \int \frac{A dx}{x+1} + \int \frac{(Bx+C) dx}{(x^2-x+1)} \quad (*), \text{ luego:}$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{(Bx+C)}{(x^2-x+1)} \Rightarrow 1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$\therefore \begin{cases} x=-1 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3} \\ x=0 \Rightarrow 1 = A+C \Rightarrow C = 1-A \Rightarrow C = \frac{2}{3} \\ x=1 \Rightarrow 1 = A+(B+C)2 \Rightarrow 1 = \frac{1}{3} + 2B + 2C \Rightarrow \frac{1}{3} = B+C \Rightarrow B = \frac{1}{3} - C \\ \Rightarrow B = -\frac{1}{3} \end{cases}$$

$$\begin{aligned} (*) &= \frac{1}{3} \int \frac{dx}{x+1} + \int \frac{(-\frac{1}{3}x + \frac{2}{3}) dx}{(x^2-x+1)} = \frac{1}{3} \ell\eta|x+1| - \frac{1}{3} \int \frac{(x-2) dx}{x^2-x+1} \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \int \frac{(2x-4) dx}{x^2-x+1} = \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \int \frac{(2x-1-3) dx}{x^2-x+1} \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \int \frac{(2x-1) dx}{x^2-x+1} + \frac{1}{2} \int \frac{dx}{x^2-x+1} \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \ell\eta|x^2-x+1| + \frac{1}{2} \int \frac{dx}{(x^2-x+\frac{1}{4}) + \frac{3}{4}} \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \ell\eta|x^2-x+1| + \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \ell\eta|x^2-x+1| + \frac{1}{2} \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arc} \tau g \frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c \\ &= \frac{1}{3} \ell\eta|x+1| - \frac{1}{6} \ell\eta|x^2-x+1| + \frac{\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x-1}{\sqrt{3}} + c \end{aligned}$$

$$= \ell \eta \left| \frac{\sqrt[3]{x+1}}{\sqrt{x^2-x+1}} \right| + \frac{\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x-1}{\sqrt{3}} + c$$

7.16.- $\int \frac{(x+5)dx}{x^2-x+6}$

Solución.-

$$\int \frac{(x+5)dx}{x^2-x+6} = \int \frac{(x+5)dx}{(x+3)(x-2)} = \int \frac{A dx}{(x+3)} + \int \frac{B dx}{(x-2)} \quad (*), \text{ luego:}$$

$$\frac{(x+5)}{(x^2+x-6)} = \frac{A}{(x+3)} + \frac{B}{(x-2)} \Rightarrow x+5 = A(x-2) + B(x+3)$$

$$\therefore \begin{cases} x=2 \Rightarrow 7 = 5B \Rightarrow B = 7/5 \\ x=-3 \Rightarrow 2 = -5A \Rightarrow A = -2/5 \end{cases}$$

$$(*) = -\frac{2}{5} \int \frac{dx}{x+3} + \frac{7}{5} \int \frac{dx}{x-2} = -\frac{2}{5} \ell \eta |x+3| + \frac{7}{5} \ell \eta |x-2| + c = \frac{1}{5} \ell \eta \left| \frac{(x-2)^7}{(x+3)^2} \right| + c$$

7.17.- $\int \frac{(x^2+1)dx}{x^3+1}$

Solución.-

$$\int \frac{(x^2+1)dx}{x^3+1} = \int \frac{(x^2+1)dx}{(x+1)(x^2-x+1)} = \int \frac{A dx}{(x+1)} + \int \frac{(Bx+C)dx}{(x^2-x+1)} \quad (*), \text{ luego:}$$

$$\frac{(x^2+1)}{x^3+1} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)} \Rightarrow x^2+1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$\therefore \begin{cases} x=-1 \Rightarrow 2 = 3A \Rightarrow A = 2/3 \\ x=0 \Rightarrow 1 = A+C \Rightarrow C = 1/3 \\ x=1 \Rightarrow 2 = A+(B+C)2 \Rightarrow B = 1/3 \end{cases}$$

$$(*) \int \frac{(x^2+1)dx}{x^3+1} = \int \frac{(x^2+1)dx}{(x+1)(x^2-x+1)} = \frac{2}{3} \int \frac{dx}{(x+1)} + \frac{1}{3} \int \frac{(x+1)dx}{(x^2-x+1)}$$

$$= \frac{2}{3} \ell \eta |x+1| + \frac{1}{3} \int \frac{\left[\frac{1}{2}(2x-1) + \frac{2}{3} \right] dx}{(x^2-x+1)} = \frac{2}{3} \ell \eta |x+1| + \frac{1}{6} \int \frac{(2x-1)dx}{(x^2-x+1)} + \frac{1}{2} \int \frac{dx}{(x^2-x+1)}$$

$$= \frac{2}{3} \ell \eta |x+1| + \frac{1}{6} \ell \eta |x^2-x+1| + \frac{1}{2} \int \frac{dx}{(x^2-x+1)}$$

$$= \frac{2}{3} \ell \eta |x+1| + \frac{1}{6} \ell \eta |x^2-x+1| + \frac{1}{2} \int \frac{dx}{(x^2-x+\frac{1}{4})+\frac{3}{4}}$$

$$= \frac{4}{6} \ell \eta |x+1| + \frac{1}{6} \ell \eta |x^2-x+1| + \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{6} \ell \eta \left| (x+1)^4 (x^2 - x + 1) \right| + \frac{1}{2} \frac{1}{\sqrt{3}} \operatorname{arc} \tau g \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{1}{6} \ell \eta \left| (x+1)^4 (x^2 - x + 1) \right| + \frac{\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x-1}{\sqrt{3}} + c$$

7.18.- $\int \frac{(x^2+6)dx}{(x-1)^2(x-2)}$

Solución.-

$$\int \frac{(x^2+6)dx}{(x-1)^2(x-2)} = \int \frac{A dx}{(x+1)} + \int \frac{B dx}{(x-1)^2} + \int \frac{C dx}{(x+2)} \quad (*), \text{ luego:}$$

$$\frac{(x^2+6)}{(x-1)^2(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x^2+6 = A(x+1) + (x+2) + B(x+2) + C(x-1)^2$$

$$\begin{cases} x=1 \Rightarrow 7 = 3B \Rightarrow B = 7/3 \\ x=-2 \Rightarrow 10 = 9C \Rightarrow C = 10/9 \\ x=0 \Rightarrow 6 = -2A + B + C \Rightarrow A = -1/9 \end{cases}$$

$$(*) = -\frac{1}{9} \int \frac{dx}{(x+1)} + \frac{7}{3} \int \frac{dx}{(x-1)^2} + \frac{10}{9} \int \frac{dx}{(x+2)} = -\frac{1}{9} \ell \eta |x-1| - \frac{7}{3} \frac{1}{x-1} + \frac{10}{9} \ell \eta |x+2| + c$$

$$= \frac{1}{9} \ell \eta \left| \frac{(x+2)^{10}}{x-1} \right| - \frac{7}{3(x-1)} + c$$

7.19.- $\int \frac{(x^2-1)dx}{(x^2+1)(x-2)}$

Solución.-

$$\int \frac{(x^2-1)dx}{(x^2+1)(x-2)} = \int \frac{Ax+B}{(x^2+1)} dx + \int \frac{C dx}{(x-2)} \quad (*), \text{ luego:}$$

$$\frac{(x^2-1)}{(x^2+1)(x-2)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-2)} \Rightarrow x^2-1 = (Ax+B)(x-2) + C(x^2+1)$$

$$\begin{cases} x=2 \Rightarrow 3 = 5C \Rightarrow C = 3/5 \\ x=0 \Rightarrow -1 = -2B + C \Rightarrow B = 4/5 \\ x=1 \Rightarrow 0 = -(A+B) + 2C \Rightarrow A = 2/5 \end{cases}$$

$$(*) = \int \frac{(2/5 x + 4/5) dx}{(x^2+1)} + \int \frac{3/5 dx}{(x-2)} = \frac{1}{5} \int \frac{2x dx}{(x^2+1)} + \frac{4}{5} \int \frac{dx}{(x^2+1)} + \frac{3}{5} \int \frac{dx}{x-2}$$

$$= \frac{1}{5} \ell \eta |x^2+1| + \frac{4}{5} \operatorname{arc} x + \frac{3}{5} \ell \eta |x-2| + c = \frac{1}{5} \ell \eta |(x^2+1)(x-2)^3| + \frac{4}{5} \operatorname{arc} x + c$$

$$7.20.- \int \frac{xdx}{x^2 - 4x - 5}$$

Solución.-

$$\int \frac{xdx}{x^2 - 4x - 5} = \int \frac{xdx}{(x+5)(x-1)} = \int \frac{Adx}{(x+5)} + \int \frac{Bdx}{(x-1)} \quad (*), \text{ luego:}$$

$$\frac{x}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} \Rightarrow x = A(x-1) + B(x+5)$$

$$\therefore \begin{cases} x=1 \Rightarrow 1 = 6B \Rightarrow B = 1/6 \\ x=-5 \Rightarrow -5 = -6A \Rightarrow A = 5/6 \end{cases}$$

$$(*) = \frac{5}{6} \int \frac{dx}{(x+5)} + \frac{1}{6} \int \frac{dx}{(x-1)} = \frac{5}{6} \ell\eta|x+5| + \frac{1}{6} \ell\eta|x-1| + c = \frac{5}{6} \ell\eta|(x+5)^5(x-1)| + c$$

$$7.21.- \int \frac{xdx}{x^2 - 2x - 3}$$

Solución.-

$$\int \frac{xdx}{x^2 - 2x - 3} = \int \frac{xdx}{(x-3)(x+1)} = \int \frac{Adx}{(x-3)} + \int \frac{Bdx}{(x+1)} \quad (*), \text{ luego:}$$

$$\frac{x}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)} \Rightarrow x = A(x+1) + B(x-3)$$

$$\therefore \begin{cases} x=-1 \Rightarrow -1 = -4B \Rightarrow B = 1/4 \\ x=3 \Rightarrow 3 = 4A \Rightarrow A = 3/4 \end{cases}$$

$$(*) = \frac{3}{4} \int \frac{dx}{(x-3)} + \frac{1}{4} \int \frac{B}{(x+1)} = \frac{3}{4} \ell\eta|x-3| + \frac{1}{4} \ell\eta|x+1| + c = \frac{1}{4} \ell\eta|(x-3)^3(x+1)| + c$$

$$7.22.- \int \frac{(x+1)dx}{x^2 + 4x - 5}$$

Solución.-

$$\int \frac{(x+1)dx}{x^2 + 4x - 5} = \int \frac{(x+1)dx}{(x+5)(x-1)} = \int \frac{Adx}{(x+5)} + \int \frac{Bdx}{(x-1)} \quad (*), \text{ luego:}$$

$$\frac{x+1}{(x^2 + 4x - 5)} = \frac{A}{(x+5)} + \frac{B}{(x-1)} \Rightarrow x+1 = A(x-1) + B(x+5)$$

$$\therefore \begin{cases} x=1 \Rightarrow 2 = 6B \Rightarrow B = 1/3 \\ x=-5 \Rightarrow 3 = -4A \Rightarrow -6A = 2/3 \end{cases}$$

$$(*) = \frac{2}{3} \int \frac{dx}{(x+5)} + \frac{1}{3} \int \frac{B}{(x-1)} = \frac{2}{3} \ell\eta|x+5| + \frac{1}{3} \ell\eta|x-1| + c = \frac{1}{3} \ell\eta|(x+5)^2(x-1)| + c$$

$$7.23.- \int \frac{x^2 dx}{x^2 + 2x + 1}$$

Solución.-

$$\int \frac{x^2 dx}{x^2 + 2x + 1} = \int \left(1 - \frac{2x+1}{x^2 + 2x + 1} \right) dx = \int dx - \int \frac{(2x+1)dx}{x^2 + 2x + 1} = \int dx - \int \frac{(2x+1)dx}{(x+1)^2}$$

$$= x - \left[\int \frac{A dx}{(x+1)} + \int \frac{B dx}{(x+1)^2} \right] (*), \text{ luego:}$$

$$\frac{2x+1}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \Rightarrow 2x+1 = A(x+1) + B$$

$$\therefore \begin{cases} x = -1 \Rightarrow -1 = B \Rightarrow B = -1 \\ x = 0 \Rightarrow 1 = A + B \Rightarrow A = 2 \end{cases}$$

$$(*) = x - \left[2 \int \frac{dx}{(x+1)} - \int \frac{dx}{(x+1)^2} \right] = x - \left[2 \ell \eta |x+1| + \frac{1}{x+1} \right] + c = x - 2 \ell \eta |x+1| - \frac{1}{x+1} + c$$

7.24.- $\int \frac{dx}{x(x+1)^2}$

Solución.-

$$\int \frac{dx}{x(x+1)^2} = \int \frac{A dx}{x} + \int \frac{B dx}{(x+1)} + \int \frac{C dx}{(x+1)^2} (*), \text{ luego:}$$

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \Rightarrow 1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$\therefore \begin{cases} x = -1 \Rightarrow 1 = -C \Rightarrow C = -1 \\ x = 0 \Rightarrow 1 = A \Rightarrow A = 1 \\ x = 1 \Rightarrow 1 = 4A + 2B + C \Rightarrow B = -1 \end{cases}$$

$$(*) = \int \frac{dx}{x} - \int \frac{dx}{(x+1)} - \int \frac{dx}{(x+1)^2} = \ell \eta |x| - \ell \eta |x+1| + \frac{1}{x+1} + c = \ell \eta \left| \frac{x}{x+1} \right| + \frac{1}{x+1} + c$$

7.25.- $\int \frac{dx}{(x+1)(x^2+1)}$

Solución.-

$$\int \frac{dx}{(x+1)(x^2+1)} = \int \frac{A dx}{x+1} + \int \frac{Bx+C}{x^2+1} dx (*), \text{ luego:}$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1)$$

$$\therefore \begin{cases} x = -1 \Rightarrow 1 = 2A \Rightarrow A = 1/2 \\ x = 0 \Rightarrow 1 = A + C \Rightarrow C = 1/2 \\ x = 1 \Rightarrow 1 = 2A + (B+C)2 \Rightarrow B = -1/2 \end{cases}$$

$$(*) = \frac{1}{2} \int \frac{dx}{(x+1)} + \int \frac{(-1/2 x + 1/2) dx}{(x^2+1)} = \frac{1}{2} \ell \eta |x+1| - \frac{1}{2} \int \frac{x-1}{(x^2+1)} dx$$

$$= \frac{1}{2} \ell \eta |x+1| - \frac{1}{4} \int \frac{2x dx}{(x^2+1)} + \frac{1}{2} \int \frac{dx}{(x^2+1)} = \frac{1}{2} \ell \eta |x+1| - \frac{1}{4} \ell \eta |x^2+1| + \frac{1}{2} \operatorname{arc} \tau g x + c$$

$$= \frac{1}{4} \ell \eta \left| \frac{(x+1)^2}{x^2+1} \right| + \frac{1}{2} \operatorname{arc} \tau g x + c$$

7.26.- $\int \frac{dx}{x(x^2+x+1)}$

Solución.-

$$\int \frac{dx}{x(x^2+x+1)} = \int \frac{A dx}{x} + \int \frac{Bx+C}{(x^2+x+1)} dx \quad (*), \text{ luego:}$$

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+x+1)} \Rightarrow 1 = A(x^2+x+1) + (Bx+C)x$$

$$\therefore \begin{cases} x=0 \Rightarrow 1=A \Rightarrow A=1 \\ x=1 \Rightarrow 1=3A+B+C \Rightarrow B+C=-2 \\ x=-1 \Rightarrow 1=A+B-C \Rightarrow B-C=0 \end{cases}$$

$$\begin{aligned} (*) &= \int \frac{dx}{x} - \int \frac{(x+1)dx}{(x^2+x+1)} = \ell \eta |x+1| - \frac{1}{2} \int \frac{(2x+2)dx}{(x^2+x+1)} \\ &= \ell \eta |x| - \frac{1}{2} \int \frac{(2x+1)+1}{(x^2+x+1)} dx = \ell \eta |x| - \frac{1}{2} \int \frac{(2x+1)dx}{(x^2+x+1)} - \frac{1}{2} \int \frac{dx}{(x^2+x+1)} \\ &= \ell \eta |x| - \frac{1}{2} \ell \eta |x^2+x+1| - \frac{1}{2} \int \frac{dx}{(x^2+x+\frac{1}{4})+\frac{3}{4}} \\ &= \ell \eta |x| - \frac{1}{2} \ell \eta |x^2+x+1| - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\ &= \ell \eta |x| - \frac{1}{2} \ell \eta |x^2+x+1| - \frac{1}{2} \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arc} \tau g \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c \\ &= \ell \eta |x| - \frac{1}{2} \ell \eta |x^2+x+1| - \frac{\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x+1}{\sqrt{3}} + c \end{aligned}$$

7.27.- $\int \frac{2x^2+5x-1}{x^3+x^2-2x} dx$

Solución.-

$$\int \frac{(2x^2+5x-1)dx}{(x^3+x^2-2x)} = \int \frac{A dx}{x} + \int \frac{B dx}{(x-1)} + \int \frac{C dx}{(x+2)} \quad (*), \text{ luego:}$$

$$\frac{2x^2+5x-1}{(x^3+x^2-2x)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$

$$2x^2+5x-1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$$\therefore \begin{cases} x=0 \Rightarrow -1 = -2A \Rightarrow A = \frac{1}{2} \\ x=1 \Rightarrow 6 = 3B \Rightarrow B = 2 \\ x=-2 \Rightarrow -3 = 6C \Rightarrow C = -\frac{1}{2} \end{cases}$$

$$(*) = \frac{1}{2} \int \frac{dx}{x} + 2 \int \frac{dx}{(x-1)} - \frac{1}{2} \int \frac{dx}{(x+2)} = \frac{1}{2} \ell \eta |x| + 2 \ell \eta |x-1| - \frac{1}{2} \ell \eta |x+2| + c$$

$$7.28.- \int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx$$

Solución.-

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx = \int \frac{A dx}{(x-1)} + \int \frac{B dx}{(x+1)} + \int \frac{C dx}{(x+1)^2} \quad (*), \text{ luego:}$$

$$\frac{x^2 + 2x + 3}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$x^2 + 2x + 3 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 6 = 4A \Rightarrow A = \frac{3}{2} \\ x=-1 \Rightarrow 2 = -2C \Rightarrow C = -1 \\ x=0 \Rightarrow 3 = A - B - C \Rightarrow B = -\frac{1}{2} \end{cases}$$

$$(*) = \frac{3}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{3}{2} \ell \eta |x-1| - \frac{1}{2} \ell \eta |x+1| + \frac{1}{x+1} + c$$

$$= \frac{1}{2} \ell \eta \left| \frac{(x-1)^3}{x+1} \right| + \frac{1}{x+1} + c$$

$$7.29.- \int \frac{3x^2 + 2x - 2}{x^3 - 1} dx$$

Solución.-

$$\int \frac{3x^2 + 2x - 2}{x^3 - 1} dx = \int \frac{3x^2 + 2x - 2}{(x-1)(x^2 + x + 1)} dx = \int \frac{A dx}{x-1} + \int \frac{(Bx + C) dx}{(x^2 + x + 1)} \quad (*), \text{ luego:}$$

$$\frac{3x^2 + 2x - 2}{(x-1)(x^2 + x + 1)} = \frac{A}{x-1} + \frac{Bx + C}{(x^2 + x + 1)}$$

$$3x^2 + 2x - 2 = A(x^2 + x + 1) + (Bx + C)(x-1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 3 = 3A \Rightarrow A = 1 \\ x=0 \Rightarrow -2 = A - C \Rightarrow C = 3 \\ x=-1 \Rightarrow -1 = A + (-B + C)(-2) \Rightarrow B = 2 \end{cases}$$

$$(*) = \int \frac{dx}{x-1} + \int \frac{(2x+3) dx}{(x^2 + x + 1)} = \ell \eta |x-1| + \int \frac{(2x+1) + 2}{(x^2 + x + 1)} dx$$

$$= \ell \eta |x-1| + \int \frac{(2x+1) dx}{(x^2 + x + 1)} + 2 \int \frac{dx}{(x^2 + x + 1)}$$

$$= \ell \eta |x-1| + \ell \eta |x^2 + x + 1| + 2 \int \frac{dx}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \ell \eta |(x-1)(x^2+x+1)| + 2 \frac{1}{\sqrt{3}} \operatorname{arc} \tau g \frac{x+1/2}{\sqrt{3}} + c$$

$$= \ell \eta |(x-1)(x^2+x+1)| + \frac{4\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x+1}{\sqrt{3}} + c$$

7.30.- $\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2} dx$

Solución.-

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2} dx = \int \frac{A dx}{x-1} + \int \frac{(Bx+C) dx}{(x^2+2)} + \int \frac{(Dx+E) dx}{(x^2+2)^2} \quad (*), \text{ luego:}$$

$$\frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+2)} + \frac{Dx+E}{(x^2+2)^2}$$

$$x^4 - x^3 + 2x^2 - x + 2 = A(x^2+2)^2 + (Bx+C)(x-1)(x^2+2) + (Dx+E)(x-1)$$

$$= A(x^4 + 4x^2 + 4) + (Bx+C)(x^3 + 2x - x^2 - 2) + Dx^2 - Dx + Ex - E$$

$$= Ax^4 + 4Ax^2 + 4A + Bx^4 + 2Bx^2 - Bx^3 - 2Bx + Cx^3 + 2Cx - Cx^2 - 2C$$

$$\Rightarrow +Dx^2 - Dx + Ex - E$$

$$= (A+B)x^4 + (C-B)x^3 + (4A-C+2B+D)x^2 + (-2B+2C-D+E)x + (4A-2C-E)$$

Igualando coeficientes, se tiene:

$$\begin{pmatrix} A+B & & & & & =1 \\ & -B+C & & & & =-1 \\ 4A+2B & -C+D & & & & =2 \\ & -2B+2C & -D+E & & & =-1 \\ 4A & & -2C & & -E & =2 \end{pmatrix} \therefore A = \frac{1}{3}, B = \frac{2}{3}, C = -\frac{1}{3}, D = -1, E = 0$$

$$(*) = \frac{1}{3} \int \frac{dx}{x-1} + \int \frac{(\frac{2}{3}x - \frac{1}{3}) dx}{(x^2+2)} - \int \frac{xdx}{(x^2+2)^2}$$

$$= \frac{1}{3} \int \frac{dx}{x-1} + \frac{1}{3} \int \frac{2xdx}{(x^2+2)} - \frac{1}{3} \int \frac{dx}{(x^2+2)} - \frac{1}{2} \int \frac{2xdx}{(x^2+2)^2}$$

$$= \frac{1}{3} \ell \eta |x-1| + \frac{1}{3} \ell \eta |x^2+2| - \frac{\sqrt{2}}{6} \operatorname{arc} \tau g \frac{x}{\sqrt{2}} + \frac{1}{2} \frac{1}{x^2+2} + c$$

$$= \frac{1}{3} \ell \eta |(x-1)(x^2+2)| - \frac{\sqrt{2}}{6} \operatorname{arc} \tau g \frac{x}{\sqrt{2}} + \frac{1}{2(x^2+2)} + c$$

7.31.- $\int \frac{2x^2 - 7x - 1}{x^3 + x^2 - x - 1} dx$

Solución.-

$$\int \frac{2x^2 - 7x - 1}{x^3 + x^2 - x - 1} dx = \int \frac{2x^2 - 7x - 1}{(x-1)(x+1)^2} dx = \int \frac{A dx}{x-1} + \int \frac{B dx}{x+1} + \int \frac{C dx}{(x+1)^2} \quad (*), \text{ luego:}$$

$$\frac{2x^2 - 7x - 1}{(x^3 + x^2 - x - 1)} = \frac{A}{x-1} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$2x^2 - 7x - 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\therefore \begin{cases} x = -1 \Rightarrow 8 = -2C \Rightarrow C = -4 \\ x = 1 \Rightarrow -6 = 4A \Rightarrow A = -\frac{3}{2} \\ x = 0 \Rightarrow -1 = A - B - C \Rightarrow B = \frac{7}{2} \end{cases}$$

$$(*) = -\frac{3}{2} \int \frac{dx}{x-1} + \frac{7}{2} \int \frac{dx}{x+1} - 4 \int \frac{dx}{(x+1)^2} = -\frac{3}{2} \ell \eta |x-1| + \frac{7}{2} \ell \eta |x+1| + \frac{4}{x+1} + c$$

$$= -\frac{1}{2} \ell \eta \left| \frac{(x+1)^7}{(x-1)^3} \right| + \frac{4}{x+1} + c$$

7.32.- $\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} dx$

Solución.-

$$\int \frac{3x^2 + 3x + 1}{x^3 + 2x^2 + 2x + 1} dx = \int \frac{(3x^2 + 3x + 1)dx}{(x+1)(x^2 + x + 1)} = \int \frac{A dx}{x+1} + \int \frac{(Bx + C) dx}{(x^2 + x + 1)} \quad (*), \text{ luego:}$$

$$\frac{3x^2 + 3x + 1}{(x+1)(x^2 + x + 1)} = \frac{A}{x+1} + \frac{Bx + C}{(x^2 + x + 1)}$$

$$3x^2 + 3x + 1 = A(x^2 + x + 1) + (Bx + C)(x + 1)$$

$$\therefore \begin{cases} x = -1 \Rightarrow A = 1 \\ x = 0 \Rightarrow 1 = A + C \Rightarrow C = 0 \\ x = 1 \Rightarrow 7 = 3A + (B + C)(2) \Rightarrow B = 2 \end{cases}$$

$$(*) = \int \frac{dx}{x+1} + \int \frac{2x dx}{(x^2 + x + 1)} = \ell \eta |x+1| + \int \frac{(2x+1) - 1}{(x^2 + x + 1)} dx$$

$$= \ell \eta |x+1| + \int \frac{(2x+1) dx}{(x^2 + x + 1)} - \int \frac{dx}{(x^2 + x + 1)}$$

$$= \ell \eta |x+1| + \ell \eta |x^2 + x + 1| - \int \frac{dx}{(x^2 + x + \frac{1}{4}) + (\frac{\sqrt{3}}{2})^2}$$

$$= \ell \eta |x+1| + \ell \eta |x^2 + x + 1| - \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arc} \tau g \frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \ell \eta |(x+1)(x^2 + x + 1)| - \frac{2\sqrt{3}}{3} \operatorname{arc} \tau g \frac{2x+1}{\sqrt{3}} + c$$

7.33.- $\int \frac{x^3 + 7x^2 - 5x + 5}{(x-1)^2(x+1)^2} dx$

Solución.-

$$\int \frac{x^3 + 7x^2 - 5x + 5}{(x-1)^2(x+1)^3} dx = \int \frac{A dx}{x-1} + \int \frac{B dx}{(x-1)^2} + \int \frac{C dx}{x+1} + \int \frac{D dx}{(x+1)^2} + \int \frac{E dx}{(x+1)^3} \quad (*), \text{ luego:}$$

$$\frac{x^3 + 7x^2 - 5x + 5}{(x-1)^2(x+1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

$$x^3 + 7x^2 - 5x + 5 = A(x-1)(x+1)^3 + B(x+1)^3 + C(x-1)^2(x+1)^2$$

$$\Rightarrow +D(x-1)^2(x+1) + E(x-1)^2$$

$$= Ax^4 + 2Ax^3 - 2Ax - A + Bx^3 + 3Bx^2 + 3Bx + B + Cx^4 - 2Cx^2 + C$$

$$\Rightarrow +Dx^3 - Dx^2 - Dx + D + Ex^2 - 2Ex + E$$

$$= (A+C)x^4 + (2A+B+D)x^3 + (3B-2C-D+E)x^2$$

$$\Rightarrow +(-2A+3B-D-2E)x + (-A+B+C+D+E)$$

Igualando coeficientes, se tiene:

$$\begin{pmatrix} A & +C & & & = 0 \\ 2A & + B & & +D & = 1 \\ & +3B-2C- & D+ & E & = 7 \\ -2A & +3B & & -D-2E & = -5 \\ -A & + B & + C & + D & + E = 2 \end{pmatrix} \therefore A=0, B=1, C=0, D=0, E=4$$

$$(*) = \int \frac{dx}{(x-1)^2} + 4 \int \frac{dx}{(x+1)^3} = -\frac{1}{x-1} - \frac{2}{(x+1)^2} + c = -\frac{x^2 - 4x - 1}{(x-1)(x+1)^2} + c$$

7.34.- $\int \frac{2x dx}{(x^2 + x + 1)^2}$

Solución.-

$$\int \frac{2x dx}{(x^2 + x + 1)^2} = \int \frac{(Ax + B) dx}{x^2 + x + 1} + \int \frac{(Cx + D) dx}{(x^2 + x + 1)^2} \quad (*), \text{ luego:}$$

$$\frac{2x}{(x^2 + x + 1)^2} = \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{(x^2 + x + 1)^2}$$

$$2x = (Ax + B)(x^2 + x + 1) + Cx + D \Rightarrow 2x = Ax^3 + Ax^2 + Ax + Bx^2 + Bx + B + Cx + D$$

$$= Ax^3 + (A+B)x^2 + (A+B+C)x + B + D, \text{ igualando coeficientes se tiene:}$$

$$\begin{pmatrix} A & & & = 0 \\ A+B & & & = 0 \\ A+B+C & & & = 2 \\ & & +D & = 0 \end{pmatrix}$$

$$\therefore A=0, B=0, C=2, D=0$$

$$(*) = \int \frac{2x dx}{(x^2 + x + 1)}, \text{ de donde el método sugerido pierde aplicabilidad; tal como se}$$

había planteado la técnica trabajada debe ser sustituida por otra:

$$\int \frac{2x dx}{(x^2 + x + 1)} = \int \frac{(2x+1) dx}{(x^2 + x + 1)} - \int \frac{dx}{(x^2 + x + 1)^2}$$

$$= \int \frac{(2x+1)dx}{(x^2+x+1)} - \frac{16}{9} \int \frac{dx}{\left\{ \left[\frac{2}{\sqrt{3}}(x+\frac{1}{2}) \right]^2 + 1 \right\}} \quad (**)$$

sea: $u = \frac{2}{\sqrt{3}}(x + \frac{1}{2}), dx = \frac{\sqrt{3}}{2} du$, entonces:

$$(**) - \frac{1}{x^2+x+1} - \frac{16}{9} \frac{\sqrt{3}}{2} \int \frac{du}{(u^2+1)^2}, \text{trabajando la integral sustituyendo}$$

trigonométricamente:

$$= -\frac{1}{x^2+x+1} - \frac{8\sqrt{3}}{9} \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta}, \text{ya que: } u = \tau g \theta, du = \sec^2 \theta d\theta$$

$$= -\frac{1}{x^2+x+1} - \frac{8\sqrt{3}}{9} \left[\frac{1}{2} \operatorname{arc} \tau g u + \frac{1}{2} \frac{u}{(u^2+1)} \right]$$

$$= -\frac{1}{x^2+x+1} - \frac{8\sqrt{3}}{9} \left\{ \frac{1}{2} \operatorname{arc} \tau g \frac{2}{\sqrt{3}}(x+\frac{1}{2}) + \frac{\frac{2}{\sqrt{3}}(x+\frac{1}{2})}{2 \left[\frac{4}{3}(x+\frac{1}{2})^2 + 1 \right]} \right\} + c$$

$$= -\frac{1}{x^2+x+1} - \frac{8\sqrt{3}}{9} \left\{ \frac{1}{2} \operatorname{arc} \tau g \frac{2}{\sqrt{3}}(x+\frac{1}{2}) + \frac{x+\frac{1}{2}}{\sqrt{3} \left[\frac{4}{3}(x+\frac{1}{2})^2 + 1 \right]} \right\} + c$$

$$= -\frac{1}{x^2+x+1} - \frac{4\sqrt{3}}{9} \operatorname{arc} \tau g \frac{2}{\sqrt{3}}(x+\frac{1}{2}) - \frac{8}{9} \frac{(x+\frac{1}{2})}{\left[\frac{4}{3}(x+\frac{1}{2})^2 + 1 \right]} + c$$

7.35.- $\int \frac{x^2+2x+3}{x^3-x} dx$

Solución.-

$$\int \frac{x^2+2x+3}{x^3-x} dx = \int \frac{x^2+2x+3}{x(x-1)(x+1)} dx = \int \frac{A dx}{x} + \int \frac{B dx}{(x-1)} + \int \frac{C dx}{(x+1)} \quad (*), \text{ luego:}$$

$$\frac{x^2+2x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}$$

$$x^2+2x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$\begin{cases} x=0 \Rightarrow 3 = -A \Rightarrow A = -3 \\ x=-1 \Rightarrow 2 = 2C \Rightarrow C = 1 \\ x=1 \Rightarrow 6 = 2B \Rightarrow B = 3 \end{cases}$$

$$(*) = -3 \int \frac{dx}{x} + 3 \int \frac{dx}{(x-1)} + \int \frac{dx}{(x+1)} = -3\ell \eta |x| + 3\ell \eta |x-1| + \ell \eta |x+1| + c$$

$$= \ell \eta \left| \frac{(x-1)^3(x+1)}{x^3} \right| + c$$

$$7.36.- \int \frac{(2x^2 - 3x + 5)dx}{(x+2)(x-1)(x-3)}$$

Solución.-

$$\int \frac{2x^2 - 3x + 5}{(x+2)(x-1)(x-3)} dx = \int \frac{A dx}{(x+2)} + \int \frac{B dx}{(x-1)} + \int \frac{C dx}{(x-3)} \quad (*), \text{ luego:}$$

$$\frac{2x^2 - 3x + 5}{(x+2)(x-1)(x-3)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x-3}$$

$$2x^2 - 3x + 5 = A(x-1)(x-3) + B(x+2)(x-3) + C(x+2)(x-1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 4 = -6B \Rightarrow B = -2/3 \\ x=3 \Rightarrow 14 = 10C \Rightarrow C = 7/5 \\ x=-2 \Rightarrow 19 = 15A \Rightarrow A = 19/15 \end{cases}$$

$$(*) = \frac{19}{15} \int \frac{dx}{x+2} - \frac{2}{3} \int \frac{dx}{x-1} + \frac{7}{5} \int \frac{dx}{x-3} = \frac{19}{15} \ell \eta |x+2| - \frac{2}{3} \ell \eta |x-1| + \frac{7}{5} \ell \eta |x-3| + c$$

$$7.37.- \int \frac{3x^2 + x - 2}{(x-1)(x^2 + 1)} dx$$

Solución.-

$$\int \frac{3x^2 + x - 2}{(x-1)(x^2 + 1)} dx = \int \frac{A dx}{(x-1)} + \int \frac{(Bx + C) dx}{(x^2 + 1)} \quad (*), \text{ luego:}$$

$$\frac{3x^2 + x - 2}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 1}$$

$$3x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(x-1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 2 = 2A \Rightarrow A = 1 \\ x=0 \Rightarrow -2 = A - C \Rightarrow C = 3 \\ x=2 \Rightarrow 12 = 5A + 2B + C \Rightarrow B = 2 \end{cases}$$

$$(*) = \int \frac{dx}{x-1} + \int \frac{(2x+3)dx}{x^2+1} = \int \frac{dx}{x-1} + \int \frac{2x dx}{x^2+1} + 3 \int \frac{dx}{x^2+1}$$

$$= \ell \eta |x-1| + \ell \eta |x^2+1| + 3 \operatorname{arc} \tau g x + c = \ell \eta |(x-1)(x^2+1)| + 3 \operatorname{arc} \tau g x + c$$

$$7.38.- \int \frac{(x+5)dx}{x^3 - 3x + 2}$$

Solución.-

$$\int \frac{(x+5)dx}{x^3 - 3x + 2} = \int \frac{(x+5)dx}{(x-1)^2(x+2)} = \int \frac{A dx}{(x-1)} + \int \frac{B dx}{(x-1)^2} + \int \frac{C dx}{(x+2)} \quad (*), \text{ luego:}$$

$$\frac{x+5}{x^3 - 3x + 2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x+5 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\therefore \begin{cases} x=1 \Rightarrow 6=3B \Rightarrow B=2 \\ x=-2 \Rightarrow 3=9C \Rightarrow C=1/3 \\ x=0 \Rightarrow 5=-2A+B+C \Rightarrow A=-1/3 \end{cases}$$

$$(*) = -\frac{1}{3} \int \frac{dx}{(x-1)} + 2 \int \frac{dx}{(x-1)^2} + \frac{1}{3} \int \frac{dx}{(x+2)} = -\frac{1}{3} \ell \eta |x-1| - \frac{2}{x-1} + \frac{1}{3} \ell \eta |x+2| + c$$

$$= \frac{1}{3} \ell \eta \left| \frac{x+2}{x-1} \right| - \frac{2}{x-1} + c$$

7.39. - $\int \frac{2x^3 + 3x^2 + x - 1}{(x+1)(x^2 + 2x + 2)^2} dx$

Solución.-

$$\int \frac{(2x^3 + 3x^2 + x - 1)dx}{(x+1)(x^2 + 2x + 2)^2} = \int \frac{A dx}{x+1} + \int \frac{(Bx+C)dx}{(x^2 + 2x + 2)} + \int \frac{(Dx+E)dx}{(x^2 + 2x + 2)^2} \quad (*), \text{ luego:}$$

$$\frac{2x^3 + 3x^2 + x - 1}{(x+1)(x^2 + 2x + 2)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x^2 + 2x + 2)} + \frac{Dx+E}{(x^2 + 2x + 2)^2}$$

$$2x^3 + 3x^2 + x - 1 = A(x^2 + 2x + 2)^2 + (Bx+C)(x^2 + 2x + 2)(x+1) + (Dx+E)(x+1)$$

$$= Ax^4 + 4Ax^3 + 8Ax^2 + 8Ax + 4A + Bx^4 + 3Bx^3 + 4Bx^2 + 2Bx + Cx^3 + 3Cx^2 + 4Cx$$

$$\Rightarrow +2C + Dx^2 + Dx + Ex + E$$

$$= (A+B)x^4 + (4A+3B+C)x^3 + (+8A+4B+3C+D)x^2$$

$$\Rightarrow +(8A+2B+4C+D+E)x + (4A+2C+E)$$

Igualando coeficientes, se tiene:

$$\begin{pmatrix} A + B & & & & = 0 \\ 4A + 3B & + C & & & = 2 \\ 8A + 4B & + 3C & + D & & = 3 \\ 8A + 2B & + 4C & + D + E & & = 1 \\ 4A & & + 2C & + E & = -1 \end{pmatrix} \therefore A = -1, B = 1, C = 3, D = -2, E = -3$$

$$(*) = -\int \frac{dx}{x+1} + \int \frac{(x+3)dx}{(x^2 + 2x + 2)} - \int \frac{(2x+3)dx}{(x^2 + 2x + 2)^2}$$

$$= -\ell \eta |x-1| + \frac{1}{2} \int \frac{(2x+6)dx}{(x^2 + 2x + 2)} - \int \frac{(2x+2)+1dx}{(x^2 + 2x + 2)^2}$$

$$= -\ell \eta |x-1| + \frac{1}{2} \int \frac{(2x+2)+4}{(x^2 + 2x + 2)} dx - \int \frac{(2x+2)dx}{(x^2 + 2x + 2)^2} - \int \frac{dx}{(x^2 + 2x + 2)^2}$$

$$= -\ell \eta |x-1| + \frac{1}{2} \int \frac{(2x+2)dx}{(x^2 + 2x + 2)} + 2 \int \frac{dx}{(x^2 + 2x + 2)} - \int \frac{(2x+2)dx}{(x^2 + 2x + 2)^2} - \int \frac{dx}{(x^2 + 2x + 2)^2}$$

$$= -\ell \eta |x-1| + \frac{1}{2} \ell \eta |x^2 + 2x + 2| + 2 \int \frac{dx}{(x+1)^2 + 1} + \frac{1}{2} \frac{1}{x^2 + 2x + 2} - \int \frac{dx}{[(x+1)^2 + 1]^2}$$

$$\begin{aligned}
&= -\ell\eta|x-1| + \frac{1}{2}\ell\eta|x^2+2x+2| + 2\operatorname{arc} \tau g(x+1) \\
&\Rightarrow + \frac{1}{2} \frac{1}{x^2+2x+2} - \frac{1}{2} \frac{x+1}{x^2+2x+2} - \frac{1}{2} \operatorname{arc} \tau g(x+1) + c \\
&= \ell\eta \left| \frac{\sqrt{x^2+2x+2}}{x+1} \right| + \frac{3}{2} \operatorname{arc} \tau g(x+1) - \frac{1}{2} \frac{x}{x^2+2x+2} + c
\end{aligned}$$

7.40.- $\int \frac{(2x^2+3x-1)dx}{x^3+2x^2+4x+2}$

Solución.-

$$\int \frac{(2x^2+3x-1)dx}{x^3+2x^2+4x+2} = \int \frac{(2x^2+3x-1)dx}{(x+1)(x^2+2x+2)} = \int \frac{A dx}{(x+1)} + \int \frac{(Bx+C)dx}{(x^2+2x+2)} \quad (*), \text{ luego:}$$

$$\frac{(2x^2+3x-1)}{(x+1)(x^2+2x+2)} = \frac{A}{(x+1)} + \frac{(Bx+C)}{(x^2+2x+2)}$$

$$2x^2+3x-1 = A(x^2+2x+2) + (Bx+C)(x+1)$$

$$\therefore \begin{cases} x=-1 \Rightarrow -2 = A \Rightarrow A = -2 \\ x=0 \Rightarrow -1 = 2A + C \Rightarrow C = 3 \\ x=1 \Rightarrow 4 = 5A + (B+C)(2) \Rightarrow B = 4 \end{cases}$$

$$(*) = -2 \int \frac{dx}{(x+1)} + \int \frac{(4x+3)dx}{x^2+2x+2} = -2\ell\eta|x+1| + 2 \int \frac{(2x+2)-1}{x^2+2x+2} dx$$

$$= -2\ell\eta|x+1| + 2 \int \frac{(2x+2)dx}{x^2+2x+2} - 2 \int \frac{dx}{x^2+2x+2}$$

$$= -2\ell\eta|x+1| + 2\ell\eta|x^2+2x+2| - 2\operatorname{arc} \tau g(x+1) + c$$

7.41.- $\int \frac{(2x+1)dx}{3x^3+2x-1}$

Solución.-

$$\int \frac{(2x+1)dx}{3x^3-2x-1} = \int \frac{(2x+1)dx}{(x-1)(3x^2+3x+1)} = \int \frac{A dx}{(x-1)} + \int \frac{(Bx+C)dx}{(3x^2+3x+1)} \quad (*), \text{ luego:}$$

$$\frac{(2x+1)}{(3x^3-2x-1)} = \frac{A}{(x-1)} + \frac{(Bx+C)}{(3x^2+3x+1)}$$

$$2x+1 = A(3x^2+3x+1) + (Bx+C)(x-1)$$

$$\therefore \begin{cases} x=1 \Rightarrow 3 = 7A \Rightarrow A = \frac{3}{7} \\ x=0 \Rightarrow 1 = A - C \Rightarrow C = -\frac{4}{7} \\ x=-1 \Rightarrow -1 = A + (-B+C)(-2) \Rightarrow B = -\frac{9}{7} \end{cases}$$

$$(*) = \frac{3}{7} \int \frac{dx}{(x-1)} - \frac{1}{7} \int \frac{(9x+4)dx}{3x^2+3x+1} = \frac{3}{7} \ell\eta|x-1| - \frac{1}{7} \int \frac{(6x+3-\frac{1}{3})dx}{3x^2+3x+1}$$

$$\begin{aligned}
&= \frac{3}{7} \ell \eta |x-1| - \frac{3}{14} \int \frac{(6x+3)dx}{3x^2+3x+1} + \frac{1}{14} \int \frac{dx}{3x^2+3x+1} \\
&= \frac{3}{7} \ell \eta |x-1| - \frac{3}{14} \ell \eta |3x^2+3x+1| + \frac{1}{14} \int \frac{dx}{3(x+\frac{1}{2})^2 + \frac{1}{4}} \\
&= \frac{3}{7} \ell \eta |x-1| - \frac{3}{14} \ell \eta |3x^2+3x+1| + \frac{2}{7} \int \frac{dx}{12(x+\frac{1}{2})^2 + 1} \\
&= \frac{3}{7} \ell \eta |x-1| - \frac{3}{14} \ell \eta |3x^2+3x+1| + \frac{\sqrt{3}}{21} \operatorname{arc} \tau g 2\sqrt{3}(x+\frac{1}{2}) + c
\end{aligned}$$

7.42.- $\int \frac{x^4 - 2x^2 + 3x + 4}{(x-1)^3(x^2 + 2x + 2)} dx$

Solución.-

$$\int \frac{x^4 - 2x^2 + 3x + 4}{(x-1)^3(x^2 + 2x + 2)} dx = \int \frac{A dx}{(x-1)} + \int \frac{B dx}{(x-1)^2} + \int \frac{C dx}{(x-1)^3} + \int \frac{(Dx + E) dx}{(x^2 + 2x + 2)} \quad (*), \text{ luego:}$$

$$\frac{x^4 - 2x^2 + 3x + 4}{(x-1)^3(x^2 + 2x + 2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx + E}{x^2 + 2x + 2}$$

$$x^4 - 2x^2 + 3x + 4 = A(x-1)^2(x^2 + 2x + 2) + B(x-1)(x^2 + 2x + 2)$$

$$\Rightarrow +C(x^2 + 2x + 2) + (Dx + E)(x-1)^3$$

$$x^4 - 2x^2 + 3x + 4 = A(x^2 - 2x + 1)(x^2 + 2x + 2) + B(x^3 + 2x^2 + 2x - x^2 - 2x - 2)$$

$$\Rightarrow +C(x^2 + 2x + 2) + (Dx + E)(x^3 - 3x^2 + 3x - 1)$$

$$x^4 - 2x^2 + 3x + 4 = Ax^4 - Ax^2 - 2Ax + 2A + Bx^3 + Bx^2 - 2B + Cx^2 + 2Cx + 2C$$

$$\Rightarrow +Dx^4 - 3Dx^3 + 3Dx^2 - Dx + Ex^3 - 3Ex^2 + 3Ex - E$$

$$x^4 - 2x^2 + 3x + 4 = (A + D)x^4 + (B - 3D + E)x^3 + (-A + B + C + 3D - 3E)x^2$$

$$\Rightarrow +(-2A + 2C - D + 3E)x + (-2A - 2B + 2C - E)$$

Igualando coeficientes se tiene:

$$\begin{cases}
A & +D & = & 1 \\
& B & -3D & + E = 0 \\
-A & + B & + C & +3D - 3E = -2 \\
-2A & & +2C & -D + 3E = 3 \\
2A & -2B & +2C & & -E = 4
\end{cases}$$

$$\therefore A = \frac{106}{125}, B = \frac{9}{25}, C = \frac{6}{5}, D = \frac{19}{125}, E = \frac{102}{125}$$

$$(*) = \frac{106}{125} \int \frac{dx}{x+1} - \frac{9}{25} \int \frac{dx}{(x-1)^2} + \frac{6}{5} \int \frac{dx}{(x-1)^3} + \frac{1}{125} \int \frac{(19x+102)dx}{(x^2+2x+2)}$$

$$= \frac{106}{125} \ell \eta |x-1| + \frac{9}{25} \frac{1}{x-1} + \frac{6}{5} \frac{1}{(-2)(x-1)^2} + \frac{19}{125} \int \frac{(x+\frac{102}{19})dx}{(x^2+2x+2)}$$

$$\begin{aligned}
&= \frac{106}{125} \ell \eta |x-1| + \frac{9}{25(x-1)} - \frac{3}{5(x-1)^2} + \frac{19}{250} \int \frac{(2x+2) + 8\frac{14}{19}}{(x^2+2x+2)} dx \\
&= \frac{106}{125} \ell \eta |x-1| + \frac{9}{25(x-1)} - \frac{3}{5(x-1)^2} + \frac{19}{250} \ell \eta |x^2+2x+2| + \frac{\cancel{19}}{250} \frac{166}{\cancel{19}} \int \frac{dx}{(x^2+2x+1)+1} \\
&= \frac{106}{125} \ell \eta |x-1| + \frac{9}{25(x-1)} - \frac{3}{5(x-1)^2} + \frac{19}{250} \ell \eta |x^2+2x+2| + \frac{166}{250} \int \frac{dx}{(x+1)^2+1} \\
&= \frac{106}{125} \ell \eta |x-1| + \frac{9}{25(x-1)} - \frac{3}{5(x-1)^2} + \frac{19}{250} \ell \eta |x^2+2x+2| + \frac{166}{250} \operatorname{arc} \tau g(x+1) + c
\end{aligned}$$

7.43.- $\int \frac{e^t dt}{e^{2t} + 3e^t + 2}$

Solución.-

$$\int \frac{e^t dt}{e^{2t} + 3e^t + 2} = \int \frac{e^t dt}{(e^t + 2)(e^t + 1)} \quad (*), \text{ Sea: } u = e^t + 1, du = e^t dt; e^t + 2 = u + 1$$

Luego:

$$(*) \int \frac{du}{(u+1)u} = \int \frac{Adu}{(u+1)} + \int \frac{Bdu}{u} \quad (**)$$

$$\frac{1}{(u+1)u} = \frac{A}{(u+1)} + \frac{B}{u} \Rightarrow 1 = Au + B(u+1)$$

$$\therefore \begin{cases} u=0 \Rightarrow 1=B \Rightarrow B=1 \\ u=-1 \Rightarrow 1=-A \Rightarrow A=-1 \end{cases}$$

$$(**) = -\int \frac{du}{(u+1)} + \int \frac{du}{u} = -\ell \eta |u+1| + \ell \eta |u| + c = -\ell \eta |e^t + 2| + \ell \eta |e^t + 1| + c$$

$$= \ell \eta \left| \frac{e^t + 1}{e^t + 2} \right| + c$$

7.44.- $\int \frac{\operatorname{sen} \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$

Solución.-

$$\int \frac{\operatorname{sen} \theta d\theta}{\cos^2 \theta + \cos \theta - 2} = \int \frac{\operatorname{sen} \theta d\theta}{(\cos \theta + 2)(\cos \theta - 1)} \quad (*),$$

Sea: $u = \cos \theta - 1, du = -\operatorname{sen} \theta d\theta, \cos \theta + 2 = u + 3$

Luego:

$$(*) \int \frac{-du}{(u+3)u} = -\int \frac{du}{u(u+3)} = -\int \frac{Adu}{u} - \int \frac{Bdu}{u+3} \quad (**)$$

$$\frac{1}{u(u+3)} = \frac{A}{u} + \frac{B}{u+3} \Rightarrow 1 = A(u+3) + Bu$$

$$\therefore \begin{cases} u=0 \Rightarrow 1=3A \Rightarrow A=1/3 \\ u=-3 \Rightarrow 1=-3B \Rightarrow B=-1/3 \end{cases}$$

$$\begin{aligned}
 (**) &= -\frac{1}{3} \int \frac{du}{u} + \frac{1}{3} \int \frac{du}{(u+3)} = -\frac{1}{3} \ell \eta |u| + \frac{1}{3} \ell \eta |u+3| + c \\
 &= -\frac{1}{3} \ell \eta |\cos \theta - 1| + \frac{1}{3} \ell \eta |\cos \theta + 2| + c, \text{ Como: } |\cos \theta| < 1, \text{ se tiene:} \\
 &= -\frac{1}{3} \ell \eta |1 - \cos \theta| + \frac{1}{3} \ell \eta |2 + \cos \theta| + c = \frac{1}{3} \ell \eta \left| \frac{2 + \cos \theta}{1 - \cos \theta} \right| + c
 \end{aligned}$$

7.45.- $\int \frac{4x^4 - 2x^3 - x^2 + 3x + 1}{(x^3 + x^2 - x - 1)} dx$

Solución.-

$$\begin{aligned}
 \int \frac{4x^4 - 2x^3 - x^2 + 3x + 1}{(x^3 + x^2 - x - 1)} dx &= \int \left(4x - 6 + \frac{9x^2 + x - 5}{x^3 + x^2 - x - 1} \right) dx \\
 &= \int 4x dx - \int 6 dx + \int \frac{(9x^2 + x - 5) dx}{x^3 + x^2 - x - 1} = 2x^2 - 6x + \int \frac{(9x^2 + x - 5) dx}{x^3 + x^2 - x - 1} (*)
 \end{aligned}$$

Trabajando sólo la integral resultante:

$$\int \frac{(9x^2 + x - 5) dx}{x^3 + x^2 - x - 1} = \int \frac{(9x^2 + x - 5) dx}{(x+1)^2(x-1)} = \int \frac{A dx}{(x+1)} + \int \frac{B dx}{(x+1)^2} + \int \frac{C dx}{(x-1)} (**), \text{ luego:}$$

$$\frac{(9x^2 + x - 5)}{(x^3 + x^2 - x - 1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

$$= 9x^2 + x - 5 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

$$\begin{cases}
 x=1 \Rightarrow 5 = 4C \Rightarrow C = \frac{5}{4} \\
 x=-1 \Rightarrow 3 = -2B \Rightarrow B = -\frac{3}{2} \\
 x=0 \Rightarrow -5 = -A - B + C \Rightarrow A = \frac{31}{4}
 \end{cases}$$

$$(**) = \frac{31}{4} \int \frac{dx}{(x+1)} - \frac{3}{2} \int \frac{dx}{(x+1)^2} + \frac{5}{4} \int \frac{dx}{(x-1)} = \frac{31}{4} \ell \eta |x+1| + \frac{3}{2(x+1)} + \frac{5}{4} \ell \eta |x-1| + c$$

$$(*) = 2x^2 - 6x + \frac{31}{4} \ell \eta |x+1| + \frac{3}{2(x+1)} + \frac{5}{4} \ell \eta |x-1| + c$$

7.46.- $\int \frac{3x^4 dx}{(x^2 + 1)^2}$

Solución.-

$$\int \frac{3x^4 dx}{(x^2 + 1)^2} = \int \frac{3x^4 dx}{(x^4 + 2x^2 + 1)} = 3 \int \left[1 - \frac{2x^2 + 1}{(x^2 + 1)^2} \right] dx = 3 \int dx - 3 \int \frac{2x^2 + 1}{(x^2 + 1)^2} dx$$

$$= 3x - 3 \int \frac{2x^2 + 1}{(x^2 + 1)^2} dx (*)$$

Trabajando sólo la integral resultante:

$$\int \frac{(2x^2 + 1) dx}{(x^2 + 1)^2} = \int \frac{(Ax + B) dx}{(x^2 + 1)} + \int \frac{(Cx + D) dx}{(x^2 + 1)^2} (**), \text{ luego:}$$

$$\frac{(2x^2+1)}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} \Rightarrow 2x^2+1 = (Ax+B)(x^2+1) + Cx+D$$

$$\Rightarrow 2x^2+1 = Ax^3 + Ax + Bx^2 + B + Cx + D \Rightarrow 2x^2+1 = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

Igualando coeficientes: $A=0, B=2, A+C=0 \Rightarrow C=0, B+D=1 \Rightarrow D=-1$

$$(**) = 2 \int \frac{dx}{(x^2+1)} - \int \frac{dx}{(x^2+1)^2} = 2 \operatorname{arc} \tau gx - \frac{1}{2} \left(\operatorname{arc} \tau gx + \frac{x}{1+x^2} \right) + c$$

$$= \frac{3}{2} \operatorname{arc} \tau gx - \frac{x}{2(1+x^2)} + c$$

$$(*) = 3x - \frac{9}{2} \operatorname{arc} \tau gx - \frac{x}{2(1+x^2)} + c$$

$$7.47.- \int \frac{(2x^2+41x-91)dx}{x^3-2x^2-11x+12}$$

Solución.-

$$\int \frac{(2x^2+41x-91)dx}{x^3-2x^2-11x+12} = \int \frac{(2x^2+41x-91)dx}{(x-1)(x+3)(x-4)}$$

$$= \int \frac{(2x^2+41x-91)dx}{(x-1)(x+3)(x-4)} = \int \frac{A dx}{x-1} + \int \frac{B dx}{x+3} + \int \frac{C dx}{x-4} \quad (*)$$

$$\frac{(2x^2+41x-91)}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}$$

$$(2x^2+41x-91) = A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x+3)$$

$$\begin{cases} x = -3 \Rightarrow 18 - 123 - 91 = B(-4)(-7) \Rightarrow B = -7 \\ \therefore x = 4 \Rightarrow 32 + 164 - 91 = C(3)(7) \Rightarrow C = 5 \\ x = 1 \Rightarrow 2 + 41 - 91 = A(4)(-3) \Rightarrow A = 4 \end{cases}$$

$$(*) = 4 \int \frac{dx}{(x-1)} - 7 \int \frac{dx}{(x+3)} + 5 \int \frac{dx}{(x-4)} = 4\ell \eta |x-1| - 7\ell \eta |x+3| + 5\ell \eta |x-4| + c$$

$$= \ell \eta \left| \frac{(x-1)^4 (x-4)^5}{(x+3)^7} \right| + c$$

$$7.48.- \int \frac{(2x^4+3x^3-x-1)dx}{(x-1)(x^2+2x+2)^2}$$

Solución.-

$$\int \frac{2x^4+3x^3-x-1}{(x-1)(x^2+2x+2)^2} dx = \int \frac{A dx}{(x-1)} + \int \frac{(Bx+C) dx}{(x^2+2x+2)} + \int \frac{(Dx+E) dx}{(x^2+2x+2)^2} \quad (*), \text{ luego:}$$

$$\frac{2x^4+3x^3-x-1}{(x-1)(x^2+2x+2)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+2x+2)} + \frac{Dx+E}{(x^2+2x+2)^2}$$

$$2x^4+3x^3-x-1 = A(x^2+2x+2)^2 + (Bx+C)(x-1)(x^2+2x+2) + (Dx+E)(x-1)$$

$$2x^4+3x^3-x-1 = A(x^4+4x^2+4+4x^3+4x^2+8x) + B(x^4+2x^3+2x^2-x^3-2x^2-2x)$$

$$\Rightarrow +C(x^3+2x^2+2x-x^2-2x-2) + D(x^2-x) + E(x-1)$$

$$2x^4 + 3x^3 - x - 1 = (A+B)x^4 + (4A+B+C)x^3 + (8A+C+D)x^2$$

$$\Rightarrow +(8A-2B-D+E)x + (4A-2C-E)$$

Igualando coeficientes se tiene:

$$\begin{pmatrix} A + B & & & & = 2 \\ 4A + B + C & & & & = 3 \\ 8A & & + C + D & & = 0 \\ 8A & -2B & & -D + E & = -1 \\ 4A & & -2C & & -E = -1 \end{pmatrix}$$

$$\therefore A = \frac{3}{25}, B = \frac{47}{25}, C = \frac{16}{25}, D = -\frac{8}{5}, E = \frac{1}{5}$$

$$(*) = \frac{3}{25} \int \frac{dx}{x-1} + \frac{1}{25} \int \frac{(47x+16)dx}{(x^2+2x+2)} - \frac{1}{5} \int \frac{(8x-1)dx}{(x^2+2x+2)^2}$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{25} \int \frac{(x+16/47)dx}{(x^2+2x+2)} - \frac{8}{5} \int \frac{(x-1/8)dx}{(x^2+2x+2)^2}$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{50} \int \frac{(2x+2) - 62/47}{(x^2+2x+2)} dx - \frac{4}{5} \int \frac{(2x+2) - 9/4}{(x^2+2x+2)^2} dx$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{50} \int \frac{(2x+2)dx}{(x^2+2x+2)} - \frac{62}{50} \int \frac{dx}{(x^2+2x+2)} - \frac{4}{5} \int \frac{(2x+2)dx}{(x^2+2x+2)^2}$$

$$\Rightarrow + \frac{9}{5} \int \frac{dx}{(x^2+2x+2)^2}$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{50} \ell \eta |x^2+2x+2| - \frac{62}{50} \int \frac{dx}{(x+1)^2+1} + \frac{4}{5} \int \frac{1}{(x^2+2x+2)}$$

$$\Rightarrow + \frac{9}{5} \int \frac{dx}{[(x+1)^2+1]^2}$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{50} \ell \eta |x^2+2x+2| - \frac{62}{50} \operatorname{arc} \tau g(x+1) + \frac{4}{5(x^2+2x+2)}$$

$$\Rightarrow + \frac{9}{5} \left[\frac{1}{2} \operatorname{arc} \tau g(x+1) + \frac{1}{2} \frac{x+1}{x^2+2x+2} \right] + c$$

$$= \frac{3}{25} \ell \eta |x-1| + \frac{47}{50} \ell \eta |x^2+2x+2| - \frac{17}{50} \operatorname{arc} \tau g(x+1) + \frac{9x+17}{10(x^2+2x+2)} + c$$

7.49.- $\int \frac{dx}{e^{2x} + e^x - 2}$

Solución.-

$$\int \frac{dx}{e^{2x} + e^x - 2} = \int \frac{dx}{(e^x)^2 + e^x - 2} = \int \frac{dx}{\left[(e^x)^2 + e^x + \frac{1}{4} \right] - 2 - \frac{1}{4}}$$

$$= \int \frac{dx}{\left[e^x + \frac{1}{2} \right]^2 - \left(\frac{3}{2} \right)^2} \quad (*), \text{ Sea: } u = e^x + \frac{1}{2}, du = e^x dx \Rightarrow dx = \frac{du}{u - \frac{1}{2}}$$

Luego:

$$(*) \int \frac{\frac{du}{u - \frac{1}{2}}}{u^2 - \left(\frac{3}{2} \right)^2} = \int \frac{du}{(u - \frac{1}{2})(u + \frac{3}{2})(u - \frac{3}{2})} = \int \frac{Adu}{u - \frac{1}{2}} - \int \frac{Bdu}{u + \frac{3}{2}} + \int \frac{Cdu}{u - \frac{3}{2}} \quad (**)$$

$$\frac{1}{(u - \frac{1}{2})(u + \frac{3}{2})(u - \frac{3}{2})} = \frac{A}{u - \frac{1}{2}} - \frac{B}{u + \frac{3}{2}} + \frac{C}{u - \frac{3}{2}}$$

$$1 = A(u + \frac{3}{2})(u - \frac{3}{2}) - B(u - \frac{1}{2})(u - \frac{3}{2}) + C(u - \frac{1}{2})(u + \frac{3}{2})$$

$$\therefore \begin{cases} u = \frac{1}{2} \Rightarrow 1 = A(2)(-1) \Rightarrow A = -\frac{1}{2} \\ u = -\frac{3}{2} \Rightarrow 1 = B(-2)(-3) \Rightarrow B = \frac{1}{6} \\ u = \frac{3}{2} \Rightarrow 1 = C(1)(3) \Rightarrow C = \frac{1}{3} \end{cases}$$

$$(**) = -\frac{1}{2} \int \frac{du}{u - \frac{1}{2}} + \frac{1}{6} \int \frac{du}{u + \frac{3}{2}} + \frac{1}{3} \int \frac{du}{u - \frac{3}{2}}$$

$$= -\frac{1}{2} \ell \eta \left| u - \frac{1}{2} \right| + \frac{1}{6} \ell \eta \left| u + \frac{3}{2} \right| + \frac{1}{3} \ell \eta \left| u - \frac{3}{2} \right| + c$$

$$= \frac{1}{6} \ell \eta \left| \frac{(u + \frac{3}{2})(u - \frac{3}{2})^2}{(u - \frac{1}{2})^3} \right| + c = \frac{1}{6} \ell \eta \left| \frac{(e^x + 2)(e^x - 1)^2}{(e^x)^3} \right| + c = \frac{1}{6} \ell \eta \left| \frac{(e^x + 2)(e^x - 1)^2}{e^{3x}} \right| + c$$

7.50.- $\int \frac{\text{sen } x dx}{\cos x(1 + \cos^2 x)}$

Solución.-

$$\int \frac{\text{sen } x dx}{\cos x(1 + \cos^2 x)} = \int \frac{-\text{sen } x dx}{\cos x(1 + \cos^2 x)} = -\int \frac{du}{u(1 + u^2)} = -\int \frac{Adu}{u} - \int \frac{(Bu + C)du}{(1 + u^2)} \quad (*)$$

Sea: $u = \cos x, du = -\text{sen } x dx$

$$\frac{1}{u(1 + u^2)} = \frac{A}{u} + \frac{(Bu + C)}{(1 + u^2)} \Rightarrow 1 = A(1 + u^2) + (Bu + C)u$$

$$1 = A + Au^2 + Bu^2 + Cu \Rightarrow 1 = (A + B)u^2 + Cu + A$$

Igualando Coeficientes se tiene:

$$\therefore \begin{cases} A + B = 0 \Rightarrow B = -A \Rightarrow B = -(1) \Rightarrow B = -1 \\ C = 0, \\ A = 1 \end{cases}$$

$$(*) = -\int \frac{du}{u} + \int \frac{udu}{1 + u^2} = -\ell \eta |u| + \ell \eta \left| \sqrt{1 + u^2} \right| + c = -\ell \eta |\cos x| + \ell \eta \left| \sqrt{1 + (\cos x)^2} \right| + c$$

$$= \ell \eta \left| \frac{\sqrt{1 + (\cos x)^2}}{\cos x} \right| + c$$

$$7.51.- \int \frac{(2 + \tau g^2 \theta) \sec^2 \theta d\theta}{1 + \tau g^3 \theta}$$

Solución.-

$$\int \frac{(2 + \tau g^2 \theta) \sec^2 \theta d\theta}{1 + \tau g^3 \theta} = \int \frac{(2 + u^2) du}{(1 + u^3)} = \int \frac{(2 + u^2) du}{(1 + u)(u^2 - u + 1)} \quad (*)$$

Sea: $u = \tau g \theta$, $du = \tau g d\theta$

$$\int \frac{(2 + u^2) du}{(1 + u^3)} = \int \frac{A du}{(1 + u)} + \int \frac{Bu + C}{(u^2 - u + 1)}, \text{ luego:}$$

$$\frac{(2 + u^2)}{(1 + u^3)} = \frac{A}{(1 + u)} + \frac{Bu + C}{(u^2 - u + 1)} \Rightarrow (2 + u^2) = A(u^2 - u + 1) + (Bu + C)(1 + u)$$

$$(2 + u^2) = Au^2 - Au + A + Bu^2 + Bu + C + Cu$$

$$(2 + u^2) = (A + B)u^2 + (-A + B + C)u + A + C$$

Igualando Coeficientes se tiene:

$$\begin{pmatrix} A + B & = 1 \\ -A + B + C & = 0 \\ A & + C = 2 \end{pmatrix} \therefore A = 1, B = 0, C = 1$$

$$(*) = \int \frac{du}{1 + u} + \int \frac{du}{u^2 - u + 1} = \int \frac{du}{1 + u} + \int \frac{du}{(u - 1/2)^2 + (\sqrt{3}/2)^2}$$

$$= \ell \eta |1 + u| + \frac{1}{\sqrt{3}/2} \arctan \tau g \frac{u - 1/2}{\sqrt{3}/2} + c = \ell \eta |1 + u| + \frac{2}{\sqrt{3}} \arctan \tau g \frac{2u - 1}{\sqrt{3}} + c$$

$$= \ell \eta |1 + \tau g \theta| + \frac{2}{\sqrt{3}} \arctan \tau g \frac{(2\tau g \theta - 1)}{\sqrt{3}} + c$$

$$7.52.- \int \frac{(5x^3 + 2) dx}{x^3 - 5x^2 + 4x}$$

Solución.-

$$\int \frac{(5x^3 + 2) dx}{x^3 - 5x^2 + 4x} = \int \frac{(5x^3 + 2) dx}{x(x-1)(x-4)} = \int \frac{A dx}{x} + \int \frac{B dx}{(x-1)} + \int \frac{C dx}{(x-4)} \quad (*)$$

$$\frac{(5x^3 + 2)}{x(x-1)(x-4)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-4)}, \text{ Luego:}$$

$$(5x^3 + 2) = A(x-1)(x-4) + Bx(x-4) + Cx(x-1)$$

Igualando Coeficientes se tiene:

$$\therefore \begin{cases} x=0 \Rightarrow 2=4A \Rightarrow A=1/2 \\ x=1 \Rightarrow 7=-3B \Rightarrow B=-7/3 \\ x=4 \Rightarrow 322=12C \Rightarrow C=161/6 \end{cases}$$

$$\begin{aligned} (*) &= \frac{1}{2} \int \frac{dx}{x} - \frac{7}{3} \int \frac{dx}{x-1} + \frac{161}{6} \int \frac{dx}{x-4} = \frac{1}{2} \ell \eta |x| - \frac{7}{3} \ell \eta |x-1| + \frac{161}{6} \ell \eta |x-4| + c \\ &= \frac{3}{6} \ell \eta |x| - \frac{14}{3} \ell \eta |x-1| + \frac{161}{6} \ell \eta |x-4| + c = \frac{1}{6} \ell \eta \left| \frac{x^3(x-4)^{161}}{(x-1)^{14}} \right| + c \end{aligned}$$

7.53.- $\int \frac{x^5 dx}{(x^3+1)(x^3+8)}$

Solución.-

$$\begin{aligned} \int \frac{x^5 dx}{(x^3+1)(x^3+8)} &= \int \frac{x^5 dx}{(x+1)(x^2-x+1)(x+2)(x^2-2x+4)} \\ &= \int \frac{A dx}{(x+1)} + \int \frac{B dx}{(x+2)} + \int \frac{(Cx+D) dx}{(x^2-x+1)} + \int \frac{(Ex+F) dx}{(x^2-2x+4)} \quad (*), \text{ luego:} \end{aligned}$$

$$\frac{x^5}{(x^3+1)(x^3+8)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{Cx+D}{(x^2-x+1)} + \frac{Ex+F}{(x^2-2x+4)}, \text{ luego:}$$

$$x^5 = A(x+2)(x^2-x+1)(x^2-2x+4) + B(x+1)(x^2-x+1)(x^2-2x+4)$$

$$\Rightarrow +(Cx+D)(x+1)(x+2)(x^2-2x+4) + (Ex+F)(x+1)(x+1)(x^2-x+1)$$

$$x^5 = A(x^5+8x^2-x^4-8x+x^3+8) + B(x^5-2x^4+4x^3+x^2-2x+4)$$

$$\Rightarrow +(Cx+D)(x^4+8x+x^3+8) + (Ex+F)(x^4+2x^3+x+2)$$

$$x^5 = (A+B+C+E)x^5 + (-A-2B+C+D+2E+F)x^4 + (A+4B+D+2F)x^3$$

$$\Rightarrow +(8A+B+8C+E)x^2 + (-8A-2B+8C+8D+2E+F)x + (8A+4B+8D+2F)$$

Igualando coeficientes se tiene:

$$\begin{pmatrix} A + B + C + E = 1 \\ -A - 2B + C + D + 2E + F = 0 \\ A + 4B + D + 2F = 0 \\ 8A + B + 8C + E = 0 \\ 8A - 2B + 8C + 8D + 2E + F = 0 \\ 8A + 4B + 8D + 2F = 0 \end{pmatrix}$$

$$\therefore A = -1/21, B = 8/21, C = -2/21, D = 1/21, E = 16/21, F = -16/21$$

$$\begin{aligned} (*) &= -\frac{1}{21} \int \frac{dx}{x+1} + \frac{8}{21} \int \frac{dx}{(x+2)} - \frac{1}{21} \int \frac{(2x-1) dx}{(x^2-x+1)} + \frac{16}{21} \int \frac{(x-1) dx}{(x^2-2x+4)} \\ &= -\frac{1}{21} \ell \eta |x+1| + \frac{8}{21} \ell \eta |x+2| - \frac{1}{21} \ell \eta |x^2-x+1| + \frac{8}{21} \int \frac{(2x-2) dx}{x^2-2x+4} \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{21} \ell \eta |x+1| + \frac{8}{21} \ell \eta |x+2| - \frac{1}{21} \ell \eta |x^2 - x + 1| - \frac{8}{21} \ell \eta |x^2 - 2x + 4| + c \\ &= \frac{1}{21} \ell \eta \left| \frac{[(x+2)(x^2 - 2x + 4)]^8}{(x+1)(x^2 - x + 1)} \right| + c \end{aligned}$$

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