

Integral Definida

Ejercicios resueltos

4.7-1 Resuelve las siguientes integrales definidas:

a) $\int_0^2 3x^2 dx$

b) $\int_0^1 e^x dx$

c) $\int_1^e \frac{1}{x} dx$

d) $\int_{-1}^1 (x + 2x^2 - x^3 + 5x^4) dx$

e) $\int_2^3 \frac{1}{\sqrt{x-1}} dx$

f) $\int_1^2 \frac{2x+1}{x^2+x} dx$

g) $\int_0^{2\pi} \text{sen} x dx$

h) $\int_0^1 \frac{1}{1+x^2} dx$

i) $\int_2^5 \frac{1}{(x-1) \cdot (x+2)} dx$

j) $\int_0^1 \frac{x}{1+x^4} dx$

Solución

a) $\int_0^2 3x^2 dx = [x^3]_0^2 = (2^3) - (0^3) = 8 - 0 = 8$

b) $\int_0^1 e^x dx = [e^x]_0^1 = (e^1) - (e^0) = e - 1$

c) $\int_1^e \frac{1}{x} dx = [\ln|x|]_1^e = (\ln|e|) - (\ln|1|) = 1 - 0 = 1$

d) $\int_{-1}^1 (x + 2x^2 - x^3 + 5x^4) dx = \left[\frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + x^5 \right]_{-1}^1 =$

$$= \left(\frac{1}{2} + \frac{2}{3} - \frac{1}{4} + 1 \right) - \left(\frac{1}{2} - \frac{2}{3} - \frac{1}{4} - 1 \right) = \frac{2}{3} + 1 + \frac{2}{3} + 1 = \frac{4}{3} + 2 = \frac{10}{3}$$

$$e) \int_2^3 \frac{1}{\sqrt{x-1}} dx$$

$$\int \frac{1}{\sqrt{x-1}} dx = \left(\begin{array}{l} t = x-1 \\ dt = dx \end{array} \right) = \int \frac{1}{\sqrt{t}} dt = \int t^{-1/2} dt = \frac{t^{1/2}}{1/2} + C = 2\sqrt{x-1} + C$$

$$\int_2^3 \frac{1}{\sqrt{x-1}} dx = \left[2\sqrt{x-1} \right]_2^3 = (2\sqrt{2}) - (2\sqrt{1}) = 2\sqrt{2} - 2$$

$$f) \int_1^2 \frac{2x+1}{x^2+x} dx$$

$$\int \frac{2x+1}{x^2+x} dx = \int \frac{2x+1}{x \cdot (x+1)} dx = \int \frac{A}{x} dx + \int \frac{B}{x+1} dx$$

$$\frac{2x+1}{x \cdot (x+1)} = \frac{A}{x} + \frac{B}{x+1} \Rightarrow 2x+1 = A \cdot (x+1) + B \cdot x$$

$$\left. \begin{array}{l} x=0 \Rightarrow 1 = A \\ x=-1 \Rightarrow -1 = -B \end{array} \right\} \Rightarrow A=B=1$$

$$\int_1^2 \frac{2x+1}{x^2+x} dx = \int_1^2 \frac{1}{x} dx + \int_1^2 \frac{1}{x+1} dx = \left[\ln|x| + \ln|x+1| \right]_1^2 =$$

$$= (\ln|2| + \ln|3|) - (\ln|1| + \ln|2|) = \ln 2 + \ln 3 - \ln 2 = \ln 3$$

$$g) \int_0^{2\pi} \sin x dx = \left[-\cos x \right]_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = -1 + 1 = 0$$

$$h) \int_0^1 \frac{1}{1+x^2} dx = \left[\arctag x \right]_0^1 = (\arctag 1) - (\arctag 0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$i) \int_2^5 \frac{1}{(x-1) \cdot (x+2)} dx$$

$$\int \frac{1}{(x-1) \cdot (x+2)} dx = \int \frac{A}{x-1} dx + \int \frac{B}{x+2} dx$$

$$\frac{1}{(x-1) \cdot (x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \Rightarrow 1 = A \cdot (x+2) + B \cdot (x-1)$$

$$\left. \begin{array}{l} x=1 \Rightarrow 1 = 3A \\ x=-2 \Rightarrow 1 = -3B \end{array} \right\} \Rightarrow A = \frac{1}{3} \quad B = -\frac{1}{3}$$

$$\int_2^5 \frac{1}{(x-1) \cdot (x+2)} dx = \frac{1}{3} \int_2^5 \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx = \frac{1}{3} [\ln|x-1| - \ln|x+2|]_2^5 =$$

$$= \frac{1}{3} (\ln|4| - \ln|7|) - \frac{1}{3} (\ln|1| - \ln|4|) = \frac{1}{3} \ln 4 - \frac{1}{3} \ln 7 + \frac{1}{3} \ln 4 = \frac{2}{3} \ln 4 - \frac{1}{3} \ln 7$$

$$j) \int_0^1 \frac{x}{1+x^4} dx$$

$$\int \frac{x}{1+x^4} dx = \left(\begin{array}{l} t = x^2 \\ dt = 2x dx \end{array} \right) = \frac{1}{2} \int \frac{1}{1+t^2} dt = \frac{1}{2} \arctg(t) + C = \frac{1}{2} \arctg(x^2) + C$$

$$\int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} [\arctg(x^2)]_0^1 = \frac{1}{2} (\arctg(1)) - \frac{1}{2} (\arctg(0)) = \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \cdot 0 = \frac{\pi}{8}$$

4.7-2 Calcula el área de la región limitada por las siguientes gráficas:

$$a) \left. \begin{array}{l} y = x + 1 \\ y = 0 \text{ (EJE OX)} \\ x = 0 \\ x = 1 \end{array} \right\}$$

$$b) \left. \begin{array}{l} y = x^2 + 1 \\ y = 0 \text{ (EJE OX)} \\ x = 1 \\ x = 2 \end{array} \right\}$$

$$c) \left. \begin{array}{l} y = x^3 \\ y = 0 \text{ (EJE OX)} \\ x = 0 \\ x = 2 \end{array} \right\}$$

$$d) \left. \begin{array}{l} y = x^2 \\ y = -x + 2 \\ y = 0 \text{ (EJE OX)} \end{array} \right\}$$

$$e) \left. \begin{array}{l} y = x^2 - x - 2 \\ y = 0 \text{ (EJE OX)} \\ x = 0 \\ x = 1 \end{array} \right\}$$

$$f) \left. \begin{array}{l} y = \cos x \\ y = 0 \text{ (EJE OX)} \\ x = \pi/2 \\ x = 3\pi/2 \end{array} \right\}$$

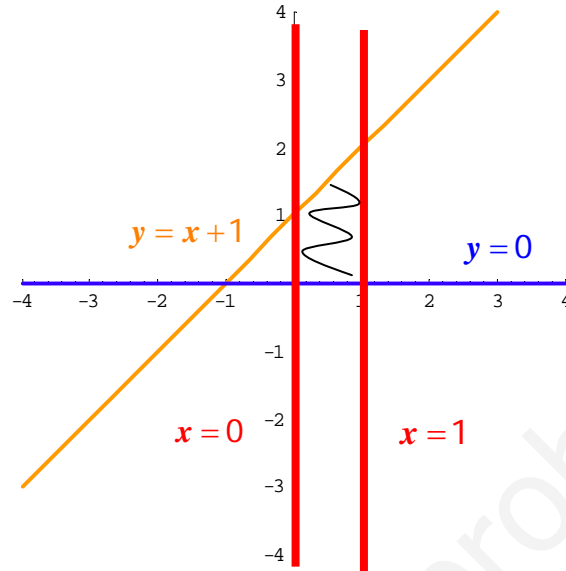
$$g) \left. \begin{array}{l} y = x^2 \\ y = x \end{array} \right\}$$

$$h) \left. \begin{array}{l} y = -x^2 + 6x \\ y = x^2 - 2x \end{array} \right\}$$

Solución

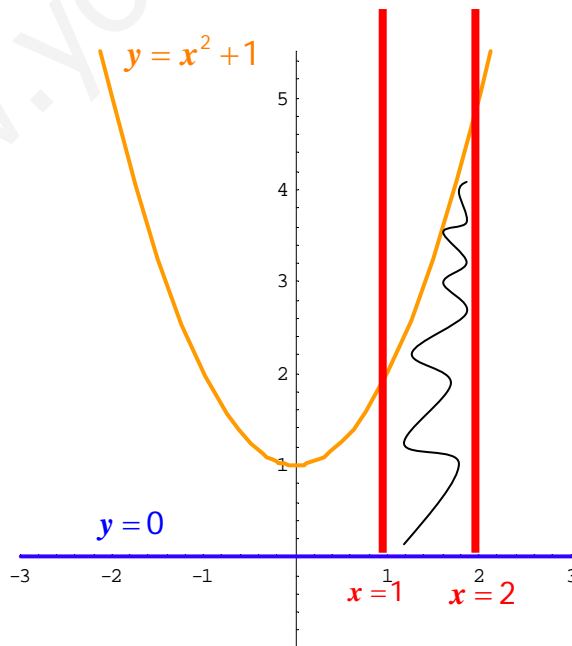
$$\left. \begin{array}{l}
 a) \ y = x + 1 \\
 y = 0 \text{ (EJE OX)} \\
 x = 0 \\
 x = 1
 \end{array} \right\}$$

$$A = \int_0^1 (x+1) dx = \left[\frac{1}{2}x^2 + x \right]_0^1 = \left(\frac{1}{2} + 1 \right) - (0) = \frac{3}{2}$$



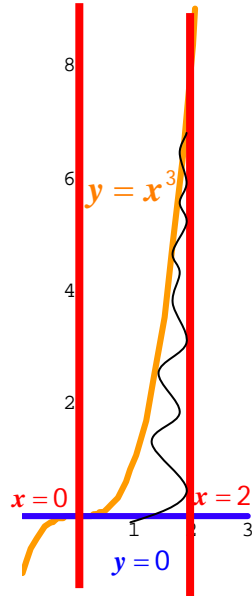
$$\left. \begin{array}{l}
 b) \ y = x^2 + 1 \\
 y = 0 \text{ (EJE OX)} \\
 x = 1 \\
 x = 2
 \end{array} \right\}$$

$$A = \int_1^2 (x^2 + 1) dx = \left[\frac{1}{3}x^3 + x \right]_1^2 = \left(\frac{8}{3} + 2 \right) - \left(\frac{1}{3} + 1 \right) = \frac{7}{3} + 1 = \frac{10}{3}$$

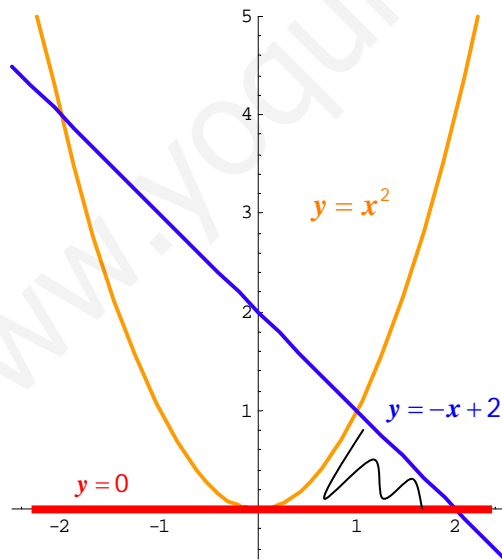


$$\left. \begin{aligned}
 c) \quad & y = x^3 \\
 & y = 0 \text{ (EJE OX)} \\
 & x = 0 \\
 & x = 2
 \end{aligned} \right\}$$

$$A = \int_0^2 x^3 dx = \left[\frac{1}{4} x^4 \right]_0^2 = \left(\frac{16}{4} \right) - 0 = 4$$



$$\left. \begin{aligned}
 d) \quad & y = x^2 \\
 & y = -x + 2 \\
 & y = 0 \text{ (EJE OX)}
 \end{aligned} \right\}$$



Puntos de corte:

$$x^2 = -x + 2 \Rightarrow x^2 + x - 2 = 0$$

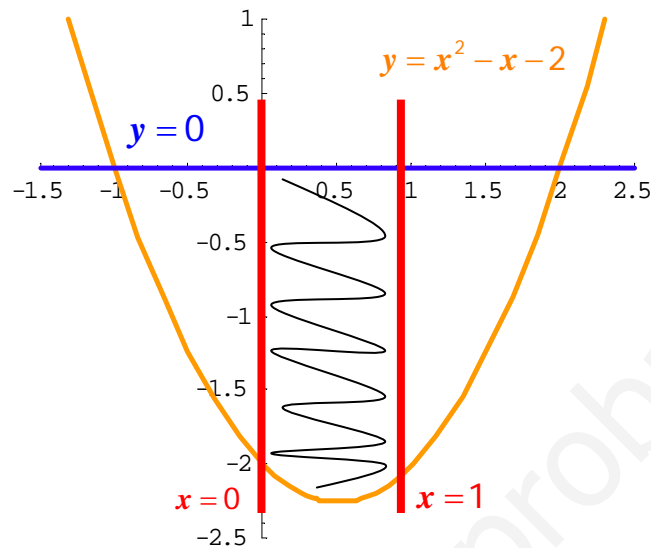
$$x = 1, -2$$

$$A = \int_0^1 x^2 dx + \int_1^2 (-x + 2) dx =$$

$$= \left[\frac{x^3}{3} \right]_0^1 + \left[-\frac{x^2}{2} + 2x \right]_1^2$$

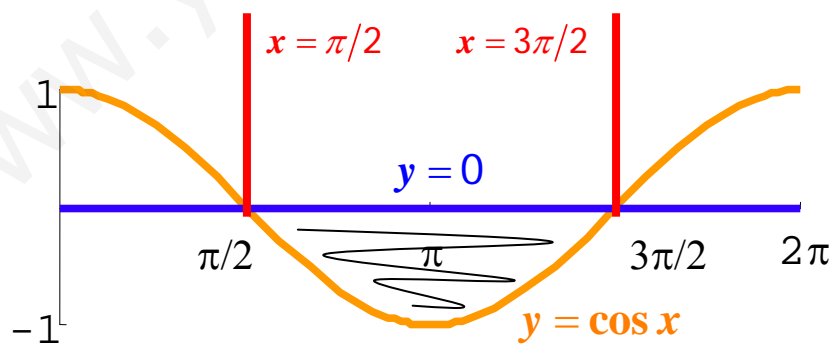
$$A = \left[\frac{1}{3} - 0 \right] + \left[(-2 + 4) - \left(-\frac{1}{2} + 2 \right) \right] = \frac{1}{3} + 2 - \frac{3}{2} = \frac{5}{6}$$

$$e) \left. \begin{array}{l} y = x^2 - x - 2 \\ y = 0 \text{ (EJE OX)} \\ x = 0 \\ x = 1 \end{array} \right\}$$



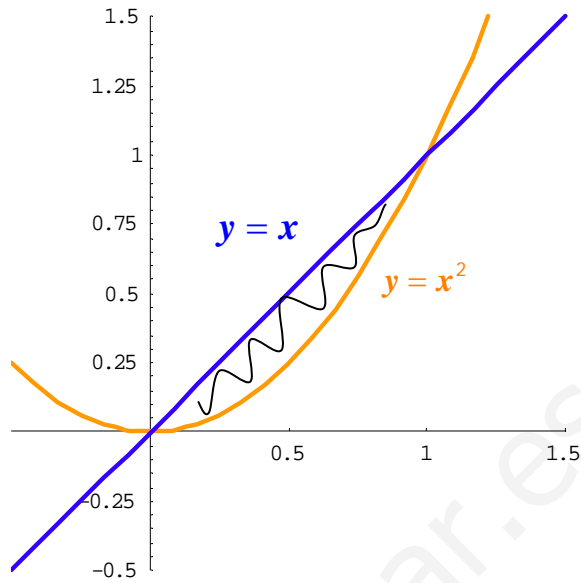
$$A = -\int_0^1 (x^2 - x - 2) dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_0^1 = -\left[\left(\frac{1}{3} - \frac{1}{2} - 2 \right) - 0 \right] = \frac{13}{6}$$

$$f) \left. \begin{array}{l} y = \cos x \\ y = 0 \text{ (EJE OX)} \\ x = \pi/2 \\ x = 3\pi/2 \end{array} \right\}$$



$$A = -\int_{\pi/2}^{3\pi/2} \cos x dx = -[\text{sen}x]_{\pi/2}^{3\pi/2} = -\left[\text{sen}\left(\frac{3\pi}{2}\right) - \text{sen}\left(\frac{\pi}{2}\right) \right] = -[-1 - 1] = 2$$

$$g) \left. \begin{array}{l} y = x^2 \\ y = x \end{array} \right\}$$



Puntos de corte:

$$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x \cdot (x - 1) = 0 \Rightarrow x = 0, 1$$

$$A = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \left[\left(\frac{1}{2} - \frac{1}{3} \right) - 0 \right] = \frac{1}{6}$$

$$h) \left. \begin{array}{l} y = -x^2 + 6x \\ y = x^2 - 2x \end{array} \right\}$$

Puntos de corte:

$$-x^2 + 6x = x^2 - 2x \Rightarrow$$

$$\Rightarrow 2x^2 - 8x = 0 \Rightarrow x \cdot (x - 4) = 0$$

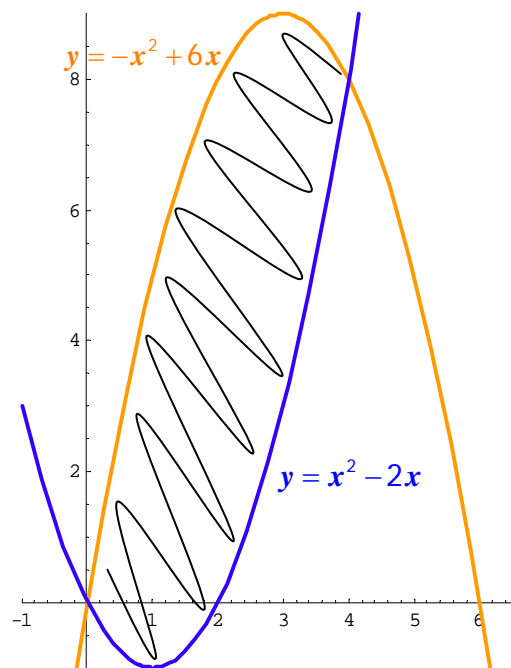
$$x = 0, 4$$

$$A = \int_0^4 [(-x^2 + 6x) - (x^2 - 2x)] dx =$$

$$= \int_0^4 (-2x^2 + 8x) dx =$$

$$= \left[-2 \frac{x^3}{3} + 8 \frac{x^2}{2} \right]_0^4 =$$

$$= \left[\left(-\frac{128}{3} + 64 \right) - 0 \right] = \frac{64}{3}$$



4.7-3 Calcula el volumen del sólido de revolución generado al girar alrededor del eje OX las siguientes gráficas:

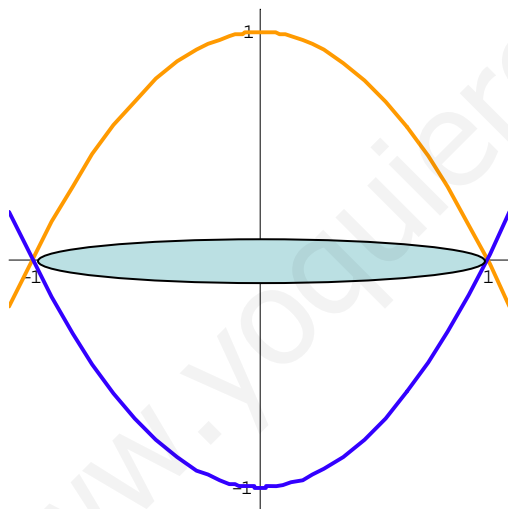
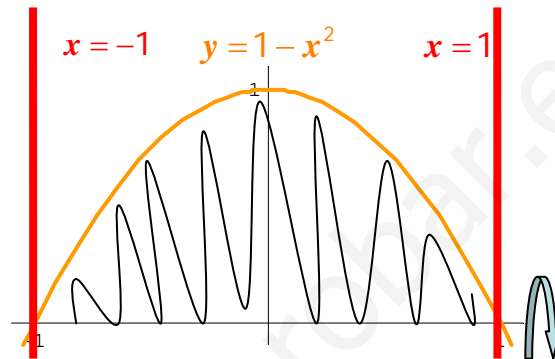
$$a) \left. \begin{array}{l} y = 1 - x^2 \\ x = -1 \\ x = 1 \end{array} \right\}$$

$$b) \left. \begin{array}{l} y = -\frac{1}{2}x^2 + 2x \\ y = \frac{1}{2}x \end{array} \right\}$$

$$c) \left. \begin{array}{l} y = 3 \\ y = 0 \\ x = 0 \\ x = 5 \end{array} \right\}$$

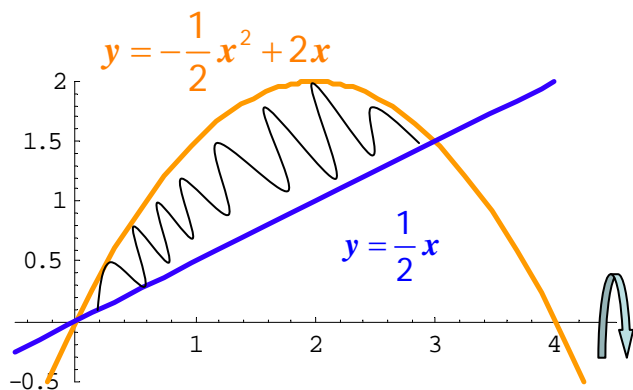
Solución

$$a) \left. \begin{array}{l} y = 1 - x^2 \\ x = -1 \\ x = 1 \end{array} \right\}$$



$$\begin{aligned} V &= \pi \int_{-1}^1 (1 - x^2)^2 dx = \\ &= \pi \int_{-1}^1 (1 - 2x^2 + x^4) dx = \\ &= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = \\ &= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \pi \left(-1 + \frac{2}{3} - \frac{1}{5} \right) = \\ &= \pi \left(2 - \frac{4}{3} + \frac{2}{5} \right) = \frac{16}{15} \pi \end{aligned}$$

$$b) \left. \begin{array}{l} y = -\frac{1}{2}x^2 + 2x \\ y = \frac{1}{2}x \end{array} \right\}$$



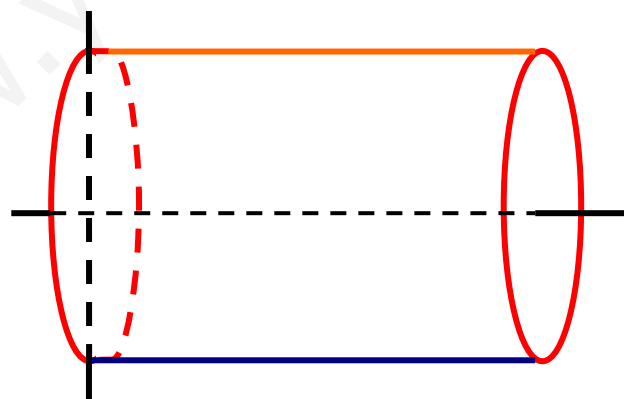
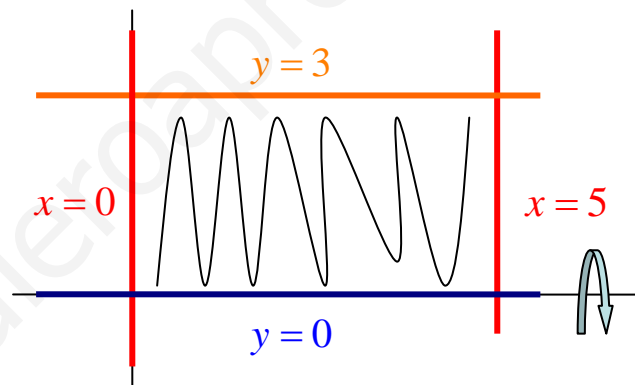
Puntos de corte:

$$-\frac{x^2}{2} + 2x = \frac{x}{2} \Rightarrow -x^2 + 4x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x \cdot (x - 3) = 0 \Rightarrow x = 0, 3$$

Se calcula el volumen generado por la gráfica de arriba (volumen lleno) y se le resta el volumen generado por la gráfica de abajo (volumen del agujero):

$$\begin{aligned} V &= \pi \int_0^3 \left(-\frac{x^2}{2} + 2x \right)^2 dx - \pi \int_0^3 \left(\frac{x}{2} \right)^2 dx = \pi \int_0^3 \left(\frac{x^4}{4} - 2x^3 + 4x^2 \right) dx - \pi \int_0^3 \frac{x^2}{4} dx = \\ &= \pi \int_0^3 \left(\frac{x^4}{4} - 2x^3 + \frac{15}{4}x^2 \right) dx = \pi \left[\frac{x^5}{20} - \frac{x^4}{2} + \frac{5x^3}{4} \right]_0^3 = \\ &= \pi \left(\frac{243}{20} - \frac{81}{2} + \frac{135}{4} \right) - \pi(0) = \frac{243 - 810 + 675}{20} \pi = \frac{108}{20} \pi = \frac{27}{5} \pi \end{aligned}$$

$$\begin{aligned} c) \quad & \left. \begin{aligned} y &= 3 \\ y &= 0 \\ x &= 0 \\ x &= 5 \end{aligned} \right\} \end{aligned}$$



$$V = \pi \int_0^5 (3)^2 dx = \pi \int_0^5 9 dx = \pi [9x]_0^5 = \pi(45 - 0) = 45\pi$$