UNIT 7: TRIGONOMETRY.

Trigonometry: Trigonometry (from Greek trigonom “triangle” and metron “measure”) is a branch of mathematics that studies triangles and the relationships between their sides and their angles.

Units of measurement of angles: There are two commonly used units of measurement of angles. The more familiar unit of measurement is **degrees**. A circle is divided into 360 equal degrees, so that a right angle is 90º.

The other common measurement for angles is **radians**.

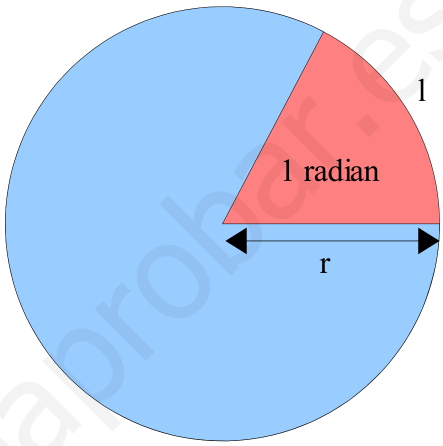
One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle.

Since the length of a circle is $2\pi r$:

$$\boxed{360^\circ = 2\pi r \text{ radians}}$$

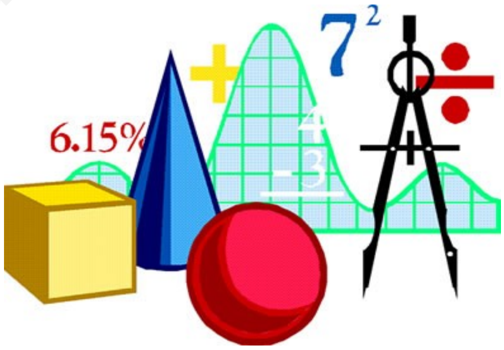
or

$$\boxed{180^\circ = \pi r \text{ radians}}$$



r = 1

Your Turn



1. Convert $\frac{\pi}{4}$ radians to degrees.

2. Convert $\frac{2\pi}{3}$ radians to degrees.

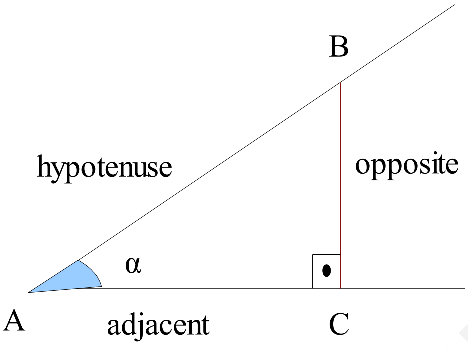
3. Convert 150º to radians.

4. Convert 135º to radians.

5. Complete the following table:

degrees	0º	30º	45º	60º	90º	180º	270º	360º	
radians									1 radian

Trigonometric ratios of an angle: When we draw a perpendicular line to one side of the angle α we get a right-angled triangle. The legs of this triangle are called **opposite** and **adjacent**.



Opposite leg is opposite the angle α, and adjacent leg is adjacent (next) to the angle α.

There are six ways to form ratios of the three sides of this triangle, and each of these ratios has a name:

sine of α = $\frac{\textit{opposite leg}}{\textit{hypotenuse}}$ $\qquad \sin\alpha = \frac{\overline{BC}}{\overline{AB}}$

cosine of α = $\frac{\textit{adjacent leg}}{\textit{hypotenuse}}$ $\qquad \cos\alpha = \frac{\overline{AC}}{\overline{AB}}$

tangent of α = $\frac{\textit{opposite leg}}{\textit{adjacent leg}}$ $\qquad \tan\alpha = \frac{\overline{BC}}{\overline{AC}}$

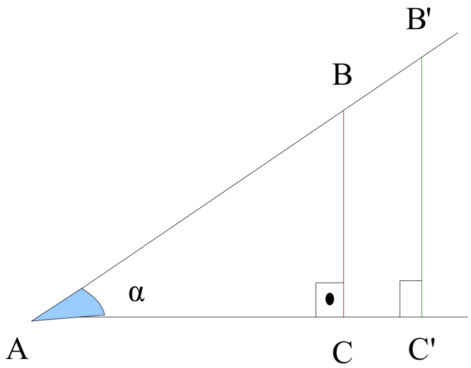
And their reciprocal (that is, the inverted fractions):

cosecant of α = $\frac{\textit{hypotenuse}}{\textit{opposite leg}}$ $\qquad csc\,\alpha = \frac{\overline{AB}}{\overline{BC}} = \frac{1}{\sin\alpha}$

secant of α = $\frac{\textit{hypotenuse}}{\textit{adjacent leg}}$ $\qquad sec\,\alpha = \frac{\overline{AB}}{\overline{AC}} = \frac{1}{\cos\alpha}$

cotangent of α = $\frac{\textit{adjacent leg}}{\textit{opposite leg}}$ $\qquad \cot\alpha = \frac{\overline{AC}}{\overline{AC}} = \frac{1}{\tan\alpha}$

If we drew other different right-angled triangles for the same angle, what would happen to the trigonometric ratios?



In ABC: $\sin\alpha=\frac{\overline{BC}}{\overline{AB}}$

In AB'C' : $\sin\alpha=\frac{\overline{B'C'}}{\overline{AB'}}$

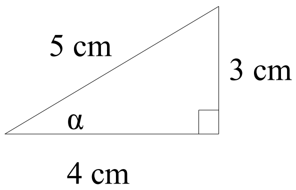
Since the triangles ABC and AB'C' are similar, $\frac{\overline{BC}}{\overline{AB}}=\frac{\overline{B'C'}}{\overline{AB'}}$.

Therefore, the value of the sine of α <u>does not depends on the right-angled triangle</u> that we use.

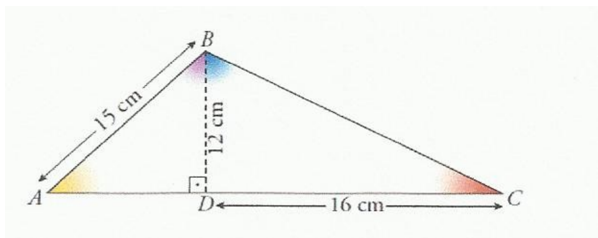
The same thing can be said about the other trigonometric ratios.

Examples:

1. Find the trigonometric ratios of the angle α:



2. Find the sine, cosine and tangent of the angles $\hat{A}$, $\hat{C}$, $\widehat{ABD}$ and $\widehat{CBD}$



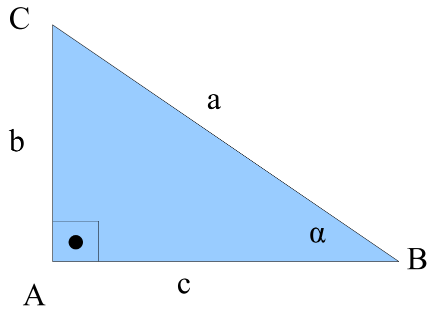
Relations between the trigonometric ratios of an angle:

The trigonometric ratios of an angle are not independent. They have same relations between them.

*Note: We use the following notation to indicate the powers of the trigonometric ratios:

$$(\sin\alpha)^2=\sin^2\alpha \qquad (\cos\alpha)^2=\cos^2\alpha \qquad (\tan\alpha)^2=\tan^2\alpha$$

The **fundamental relationships** between the trigonometric ratios of an angle are:



(1) $\boxed{\sin^2\alpha+\cos^2\alpha=1}$

(2) $\boxed{\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}}$

(3) $\boxed{1+\tan^2\alpha=\dfrac{1}{\cos^2\alpha}}$

<u>Proof:</u>

(1) $\sin^2\alpha+\cos^2\alpha=1$

(2) $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}$

(3) $1+\tan^2\alpha=\dfrac{1}{\cos^2\alpha}$

Examples:

1. If $\sin\alpha=\dfrac{4}{5}$, α an acute angle, find the values of cos α and tan α.

2. If $\cos\alpha=\frac{5}{13}$, α an acute angle, find the sin α and tan α.

3. If $\tan\alpha=2$, α an acute angle, find the sin α and cos α.

4. If $\cos\alpha=\frac{2}{3}$, α an acute angle, find the sin α and tan α.

5. If $\tan\alpha=\sqrt{5}$, α an acute angle, find the sin α and cos α.

6. Use the calculator to complete the following table:

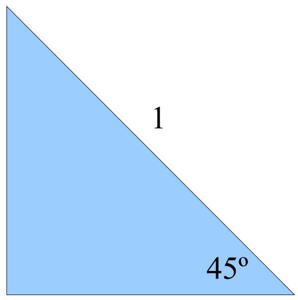
α	52º	34º23'	32º43'56"	12º23'34"
sin α				
cos α				
tan α				

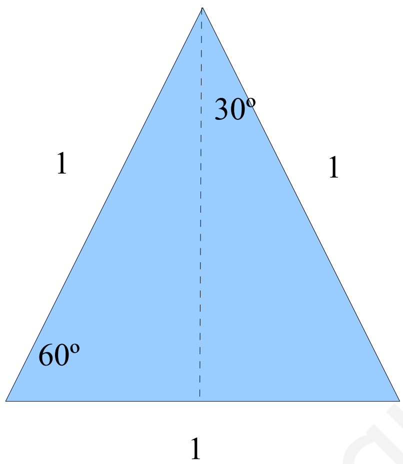
7. Find the angle α, using your calculator, in each case. Express the angles in degrees, minutes and seconds.

a) $\sin\alpha=0.58$ b) $\cos\alpha=0.75$ c) $\tan\alpha=2.5$

d) $\sin\alpha=\frac{\sqrt{5}}{3}$ e) $\cos\alpha=\frac{1}{\sqrt{3}}$ f) $\tan\alpha=1$

Trigonometric ratios of 30º, 45º and 60º:



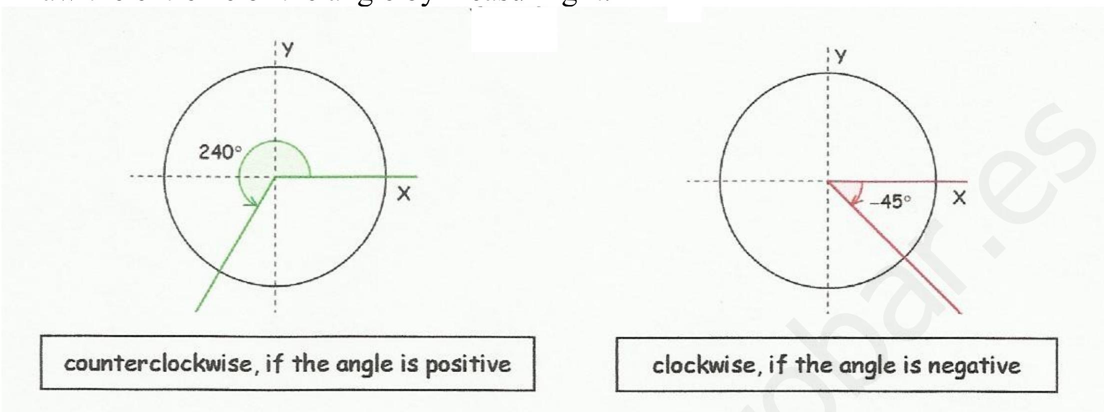


	30º	45º	60º
sin			
cos			
tan			

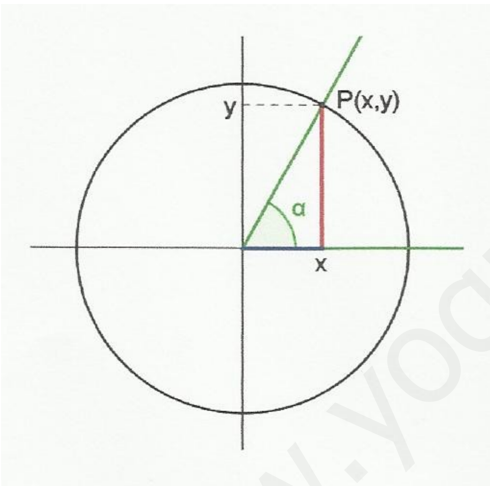
Trigonometric ratios of any angle:

Angles in a circumference: We can represent angles in a circumference which centre is the origin of coordinates:

- Take the origin of coordinates as a vertex of the triangle.
- Use the radius on the positive semi-axis of abscissas as origin of the angle.
- Draw the extreme of the angle by measuring it.



If the radius is one, the circumference is called **goniometric circumference**.



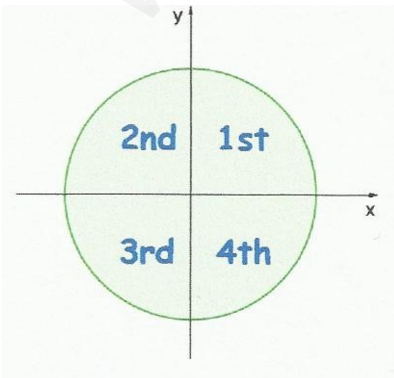
If we represent an angle α in this circumference, we get the point P(x,y). This point is the intersection of the side of the angle that is not on the x-axis with the circumference.

P(x,y) is 1 unit away from the origin. Therefore, the sine of α is the value of the y-coordinate of P, and the cosine of α is the value of the x-coordinate of P.

$\boxed{\sin\alpha = y}$ $\boxed{\cos\alpha = x}$ $\boxed{\tan\alpha = \frac{y}{x}}$

Sine, cosine and tangent in the four quadrants:

The coordinates axes divide the plane into four equal parts called **quadrants**. The angles in the goniometric circumference will be each of the four quadrants.

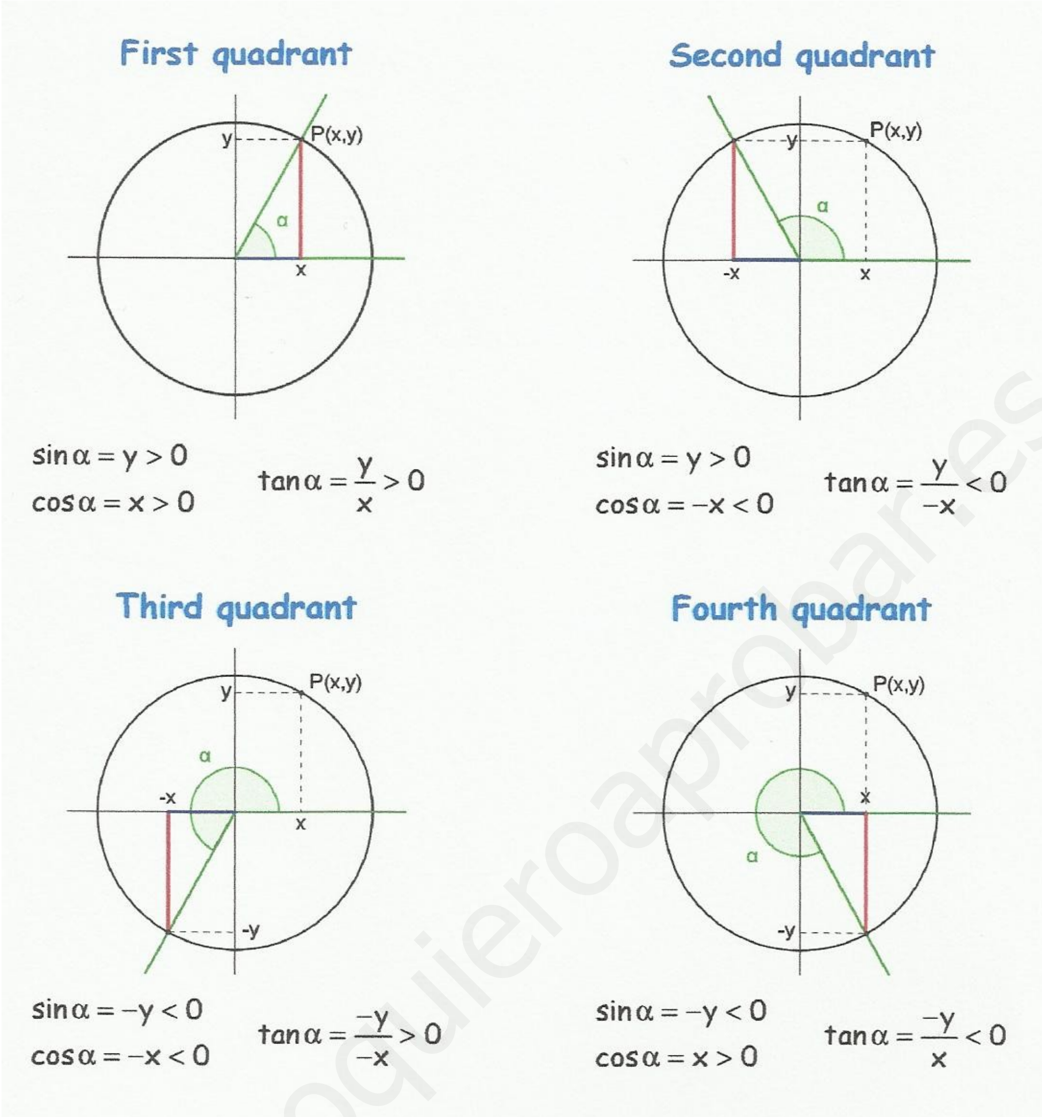


If 0º < α < 90º , α belongs to the 1st quadrant

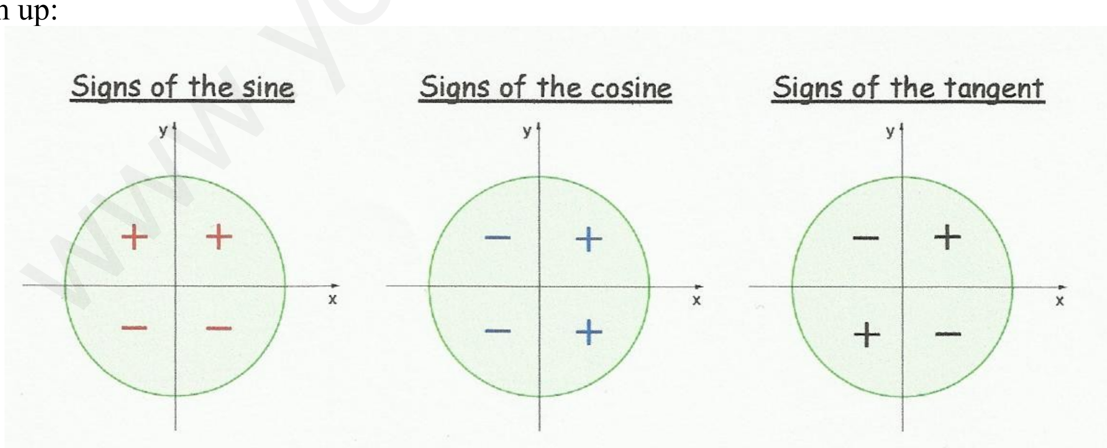
If 90º < α < 180º , α belongs to the 2nd quadrant

If 180º < α < 270º , α belongs to the 3rd quadrant

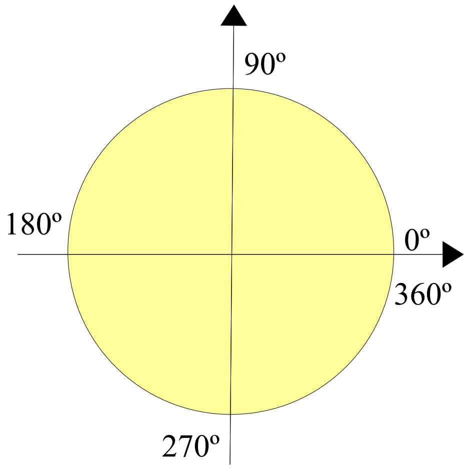
If 270º < α < 360º , α belongs to the 4th quadrant


First quadrant
P(x,y)
y
x
α
$\sin\alpha = y > 0$
$\cos\alpha = x > 0$
$\tan\alpha = \frac{y}{x} > 0$
Second quadrant
P(x,y)
y
-x
x
α
$\sin\alpha = y > 0$
$\cos\alpha = -x < 0$
$\tan\alpha = \frac{y}{-x} < 0$
Third quadrant
P(x,y)
y
-x
x
-y
α
$\sin\alpha = -y < 0$
$\cos\alpha = -x < 0$
$\tan\alpha = \frac{-y}{-x} > 0$
Fourth quadrant
P(x,y)
y
x
-y
α
$\sin\alpha = -y < 0$
$\cos\alpha = x > 0$
$\tan\alpha = \frac{-y}{x} < 0$

To sum up:

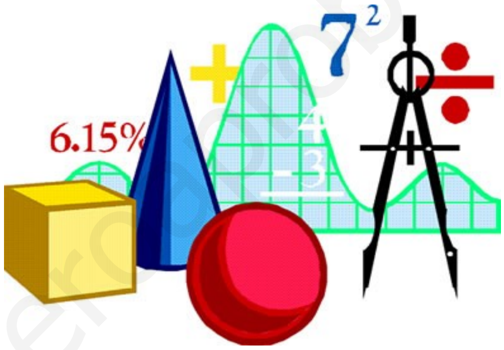

Signs of the sine
Signs of the cosine
Signs of the tangent
y
x
+
−

The trigonometric ratios of the angles that coincide on the axis: 0º, 90º, 180º, 270º and 360º are:



Angle	0º	90º	180º	270º	360º
sine					
cosine					
tangent					

Your Turn



1. If $\sin\alpha = \frac{3}{4}$, α an angle of the second quadrant, calculate cos α and tan α.

2. If $\cos\alpha = \frac{1}{4}$, α an angle of the fourth quadrant, calculate sin α and tan α.

3. If $\tan\alpha=\frac{1}{2}$, α an angle of the third quadrant, calculate sin α and cos α.

4. If $\cos\alpha=\frac{-1}{5}$, α an angle of the second quadrant, calculate sin α and tan α.

5. If $\sin\alpha=\frac{-\sqrt{3}}{3}$, α an angle of the third quadrant, calculate cos α and tan α.

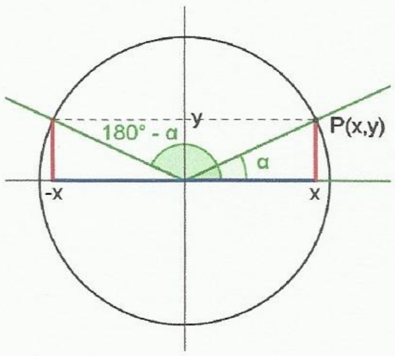
6. If $\tan\alpha=-\sqrt{2}$, α an angle of the fourth quadrant, calculate sin α and cos α.

How to reduce to the first quadrant the trigonometric ratios of any angle?

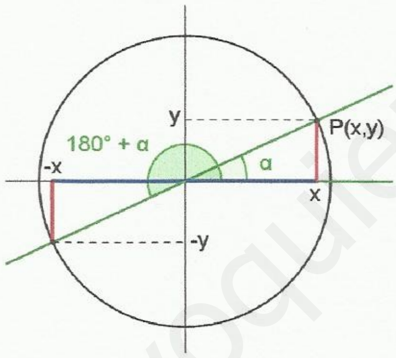
Let α be an angle in the first quadrant.

Then, 180º-α is an angle of the second quadrant, 180º+α is an angle in the third quadrant and 360º-α is an angle in the fourth quadrant.

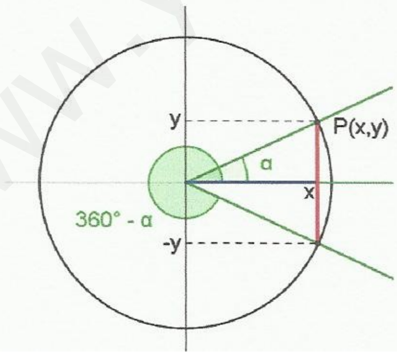
The trigonometric ratios of 180º-α, 180º+α and 360º-α can be expressed in terms of the trigonometric ratios of α.



$$\sin(180^\circ - \alpha) = \sin\alpha$$
$$\cos(180^\circ - \alpha) = -\cos\alpha$$
$$\tan(180^\circ - \alpha) = -\tan\alpha$$

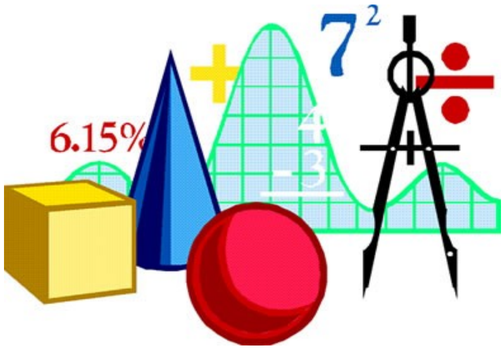


$$\sin(180^\circ + \alpha) = -\sin\alpha$$
$$\cos(180^\circ + \alpha) = -\cos\alpha$$
$$\tan(180^\circ + \alpha) = \tan\alpha$$



$$\sin(360^\circ - \alpha) = -\sin\alpha$$
$$\cos(360^\circ - \alpha) = \cos\alpha$$
$$\tan(360^\circ - \alpha) = -\tan\alpha$$

Your Turn



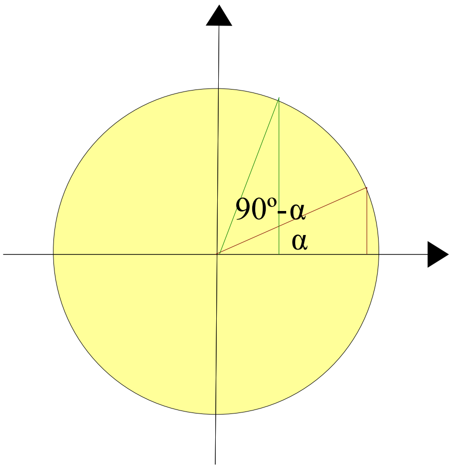
1. If α is an acute angle and $\cos\alpha=\frac{5}{9}$ what are the trigonometric ratios of the angle 180º+α?

2. Use the trigonometric ratios of 45º to find the trigonometric ratios of 135º, 225º and 315º.

3. Use the trigonometric ratios of 30º to find the trigonometric ratios of 150º, 210º and 330º.

4. Use the trigonometric ratios of 60º to find the trigonometric ratios of 120º, 240º and 300º.

Trigonometric ratios of complementary angles: α and 90º-α.



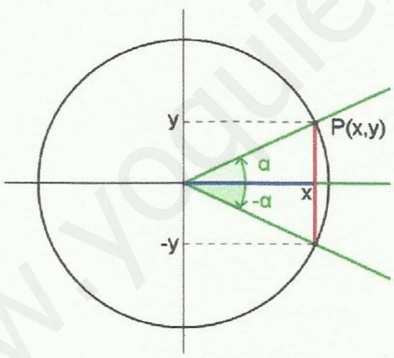
sin (90º-α) = cos α

cos (90º-α) = sin α

tan (90º-α) = cotan α

Example: If sin 20º = 0,342, calculate without using calculator, the trigonometric ratios of the angle of 70º.

Trigonometric ratios of negative angles:

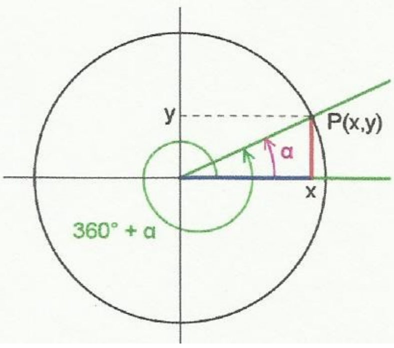


$$\sin(-\alpha) = -\sin\alpha$$

$$\cos(-\alpha) = \cos\alpha$$

$$\tan(-\alpha) = -\tan\alpha$$

Trigonometric ratios of angles greater than 360º:



$$\sin(360° + \alpha) = \sin\alpha$$

$$\cos(360° + \alpha) = \cos\alpha$$

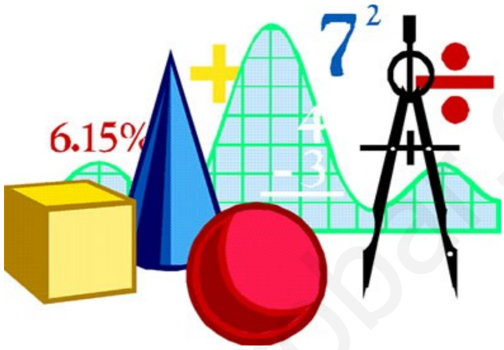
$$\tan(360° + \alpha) = \tan\alpha$$

In general, the trigonometric ratios of $n\cdot 360^{o}+\alpha$, where $n\in\mathbb{Z}$ and $0^{o}\leqslant\alpha<360^{o}$, are the same as the trigonometric ratios of α.

Example: $\sin 1500^{o} = \sin(4\cdot 360^{o}+60^{o}) = \sin 60^{o} = \frac{\sqrt{3}}{2}$

1500º	360º
60º	4 rotations

Your Turn



1. Without using calculator, find the values of these trigonometric ratios:

a) sin 120º b) cos 150º c) tan 120º

d) sin (-60º) e) cos 1125º f) tan 930º

g) sin 210º h) cos (-330º) i) tan (-315º)

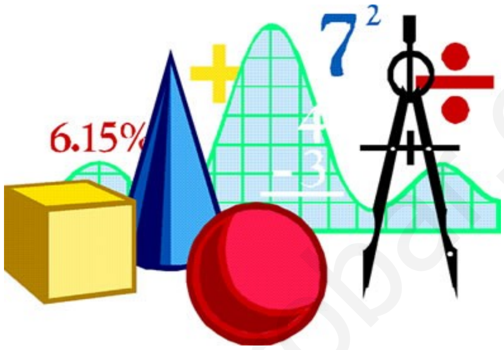
j) sin (-330º) k) cos (-1305º) l) tan (210º)

Solving the triangle:

What does "solving" the triangle means? It means that if we are given some facts about a triangle, we can find some or all the rest. For example, if we know two sides of a right-angled triangle, we can find the third side using Pythagoras' Theorem.

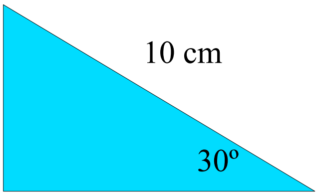
To completely solve a triangle it usually means finding everything about it, all three sides and all three angles.

Your Turn

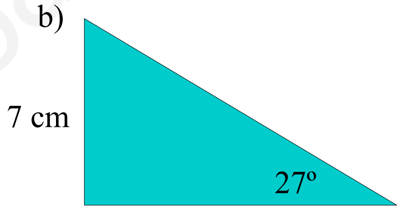


1. Solve the following triangles:

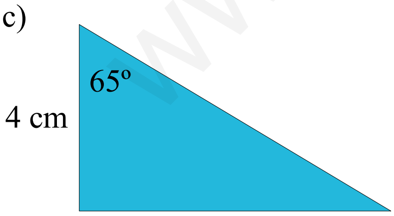
a)



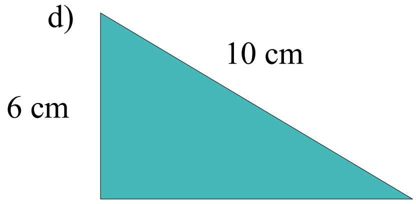
b)



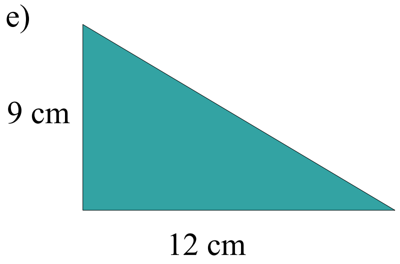
c)



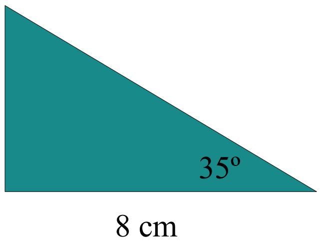
d)



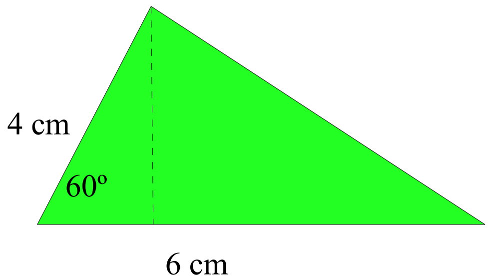
e)



f)

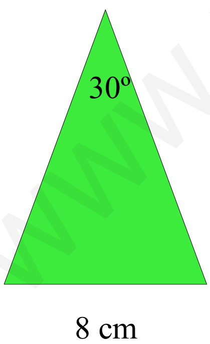


2. Calculate the area of the triangle:

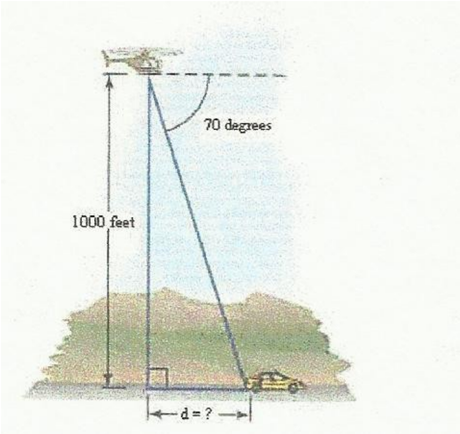


3. Calculate the area of a regular pentagon whose side is 10 cm.

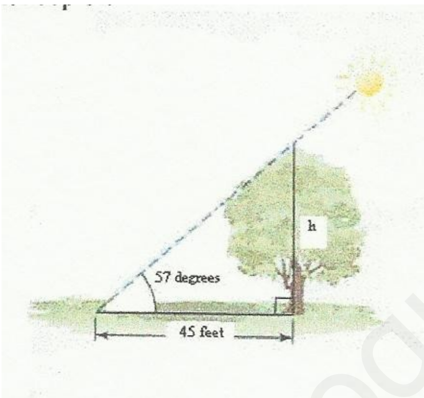
4. Calculate the perimeter and the area of an isosceles triangle whose base is 8 cm and its opposite angle is 30º.



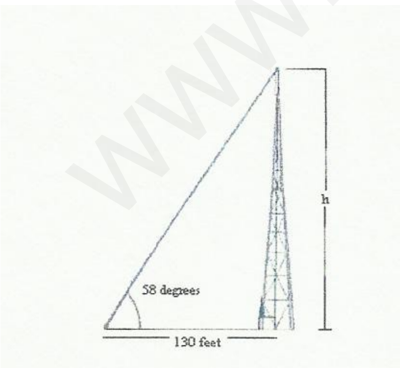
5. The picture below shows a police helicopter chasing a stolen car. The sighted angle of depression is 70 degrees. How far is the helicopter from the car? Find the distance from the stolen car to a point directly below the helicopter?



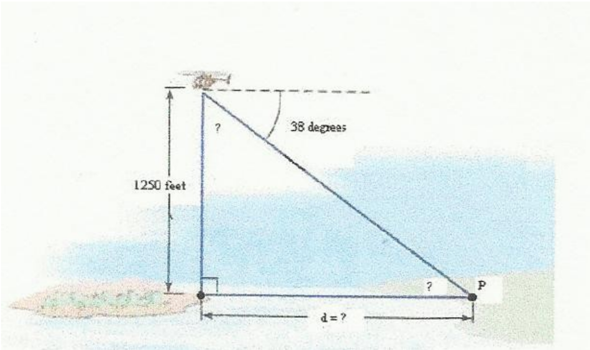
6. At a certain day, the angle of elevation of the sun is 57 degrees. Find the height of the tree if its shadow length is 45 feet. What is the distance from the top of the tree to the point in the ground made by the shadow?



7. The figure below shows a 58 degrees angle of elevation from a point on the ground that is 130 feet from the base of a radio tower to its top. How tall is the radio tower?



8. The helicopter in the picture is hovering 1250 feet above a small island. The angle of depression from the helicopter to the point P on the mainland is 38 degrees. How far is the island from the point P on the mainland? How far is the helicopter from the point P on the mainland? What are the degree measures of the two "?" shown in the picture?



9. The tallest television-transmitting tower in the world is in the state of North Dakota. From a point on level ground one mile from the base of the tower, the angle of elevation is 21 degrees. To the nearest foot, how high is this tower?

10. A balloon is hovering 800 ft. above a lake. The balloon is observed by the crew of a boat as they look upwards at an angle of 20 degrees. Twenty-five seconds later, the crew has to look at an angle of 65 degrees to see the balloon. How faster was the boat travelling?

11. Village A is 15 km away from Village B. From both villages, you can see a balloon in the air. The angle of elevation from village A to the balloon is 30º and the angle of elevation from Village B to the balloon is 20º. How high in the air is the balloon?

12. Maria and Anthony are standing 5 metres apart and are looking up at the top of a tree. Maria is closer to the tree than Anthony. The angle of elevation from where Maria is standing is 45º. The angle of elevation from where Anthony is standing to the top of the tree is 30º. How tall is the tree? How far away from the tree is each person standing?

Keywords:

Trigonometry = Trigonometría
angle = ángulo
degree = grado
radian = radián
trigonometric ratios = razones trigonométircas
opposite leg = cateto opuesto
adjacent leg = cateto adyacente, cateto contiguo
hypotenuse = hipotenusa
sine = seno
cosine = coseno
tangent = tangente
cosecant = cosecante
secant = secante
cotangent = cotangente
goniometric circumference = circunferencia goniométrica, circunferencia unidad
quadrant = cuadrante