INTEGRACION DE FUNCONES IRRACIONALES

En el caso de que el integrando contiene potencias faccionarias de la variable de integración, estas se simplifican usando una sustitución del tipo:
$x = t^n, \sqrt[n]{x} = t$**,** siendo "n "el m.c.m de los denominadores de los exponentes.

EJERCICIOS DESARROLLADOS

9.1.-Encontrar: $\int \frac{\sqrt{x}dx}{1+x}$

Solución.- La única expresión "irracional" es $\sqrt{x}$, por lo tanto:
$\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$, luego:

$$\int \frac{\sqrt{x}dx}{1+x} = \int \frac{t(2tdt)}{1+t^2} = 2\int \frac{t^2 dt}{1+t^2} = 2\int \left(1 - \frac{1}{1+t^2}\right)dt = 2\int dt - 2\int \frac{dt}{t^2+1} = 2t - 2\operatorname{arc}\tau g t + c$$

Dado que: $t = \sqrt{x}$, se tiene: $= 2\sqrt{x} - 2\operatorname{arc}\tau g\sqrt{x} + c$

Respuesta: $\int \frac{\sqrt{x}dx}{1+x} = 2\sqrt{x} - 2\operatorname{arc}\tau g\sqrt{x} + c$

9.2.-Encontrar: $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

Solución.- Análogamente al caso anterior: $\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$, luego:

$$\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = \int \frac{2\cancel{t}dt}{\cancel{t}(1+t)} = \int \frac{2dt}{1+t} = 2\ell\eta|t+1| + c$$

Dado que: $t = \sqrt{x}$, se tiene: $= 2\ell\eta\left|\sqrt{x}+1\right| + c$

Respuesta: $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\ell\eta\left|\sqrt{x}+1\right| + c$

9.3.-Encontrar: $\int \frac{dx}{3+\sqrt{x+2}}$

Solución.- La expresión "irracional" es ahora $\sqrt{x+2}$, por lo tanto:
$\sqrt{x+2} = t \Rightarrow x = t^2 - 2, dx = 2tdt$, luego:

$$\int \frac{dx}{3+\sqrt{x+2}} = \int \frac{2tdt}{3+t} = 2\int \left(1 - \frac{3}{t+3}\right)dt = 2\int dt - 6\int \frac{dt}{t+3} = 2t - 6\ell\eta|t+3| + c$$

Dado que: $t = \sqrt{x+2}$, se tiene: $= 2\sqrt{x+2} - 6\ell\eta\left|\sqrt{x+2}+3\right| + c$

Respuesta: $\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6\ell\eta\left|\sqrt{x+2}+3\right| + c$

9.4.-Encontrar: $\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}dx$

Solución.- La expresión “irracional” es ahora $\sqrt{3x+2}$, por lo tanto:

$\sqrt{3x+2}=t \Rightarrow 3x=t^2-2, dx=\frac{2}{3}tdt$, luego:

$$\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}dx=\int \frac{1-t}{1+t}\frac{2}{3}tdt=\frac{2}{3}\int \frac{t-t^2}{1+t}dt=\frac{2}{3}\int \left(-t+2-\frac{2}{t+1}\right)dt$$

$$=-\frac{2}{3}\int tdt+\frac{4}{3}\int dt-\frac{4}{3}\int \frac{dt}{t+1}=-\frac{1}{3}t^2+\frac{4}{3}t-\frac{4}{3}\ell\eta\left|t+1\right|+c$$

Dado que: $t=\sqrt{3x+2}$, se tiene:

$$=-\frac{1}{3}(3x+2)+\frac{4}{3}\sqrt{3x+2}-\frac{4}{3}\ell\eta\left|\sqrt{3x+2}+1\right|+c$$

$$=-x-\frac{2}{3}+\frac{4}{3}\sqrt{3x+2}-\frac{4}{3}\ell\eta\left|\sqrt{3x+2}+1\right|+c=-x-\frac{2}{3}+\frac{4}{3}\left(\sqrt{3x+2}-\ell\eta\left|\sqrt{3x+2}+1\right|\right)+c$$

Respuesta: $\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}dx=-x-\frac{2}{3}+\frac{4}{3}\left(\sqrt{3x+2}-\ell\eta\left|\sqrt{3x+2}+1\right|\right)+c$

9.5.- Encontrar: $\int \sqrt{1+\sqrt{x}}dx$

Solución.- La expresión “irracional” es ahora $\sqrt{x}$, por lo tanto:

$\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$, luego: $\int (\sqrt{1+\sqrt{x}})dx=\int \sqrt{1+t}\,2tdt$, como apareció la expresión: $\sqrt{1+t}$; se procede análogamente: $w=\sqrt{1+t} \Rightarrow t=w^2-1, dt=2wdw$, esto es: $\sqrt{1+t}\,2tdt=\int w2(w^2-1)2wdw=4\int (w^4-w^2)dw=\frac{4w^5}{5}-\frac{4w^3}{3}+c$

Dado que: $w=\sqrt{1+t}$, se tiene: $=\frac{4(1+t)^{5/2}}{5}-\frac{4(1+t)^{3/2}}{3}+c$

Respuesta: $\int \sqrt{1+\sqrt{x}}dx=\frac{4(1+\sqrt{x})^{5/2}}{5}-\frac{4(1+\sqrt{x})^{3/2}}{3}+c$

9.6.-Encontrar: $\int \frac{dx}{\sqrt{x+1}+\sqrt[4]{x+1}}$

Solución.- Previamente se tiene que el m.c.m. de los índices de Las raíces es: 4, por lo cual: $x+1=t^4, dx=4t^3dt$, de donde:

$$\int \frac{dx}{\sqrt{x+1}+\sqrt[4]{x+1}}=\int \frac{4t^3dt}{t^2+t}=4\int \left(t-1+\frac{t}{t^2+t}\right)dt=4\int tdt-4\int dt+4\int \frac{dt}{t+1}$$

$=2t^2-4t+4\ell\eta\left|t+1\right|+c$, dado que: $t=\sqrt[4]{x+1}$

Se tiene: $=2(x+1)^{1/2}-4(x+1)^{1/2}+4\ell\eta\left|(x+1)^{1/2}+1\right|+c$

Respuesta: $\int \frac{dx}{\sqrt{x+1}+\sqrt[4]{x+1}}=2(x+1)^{1/2}-4(x+1)^{1/2}+4\ell\eta\left|(x+1)^{1/2}+1\right|+c$

9.7.-Encontrar: $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Solución.- Previamente se tiene que el m.c.m. de los índices de Las raíces es: 6, por lo cual: $x = t^6 \Rightarrow t = \sqrt[6]{x}, dx = 6t^5 dt$, de donde:

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6t^5 dt}{t^3+t^2} = 6\int \frac{t^3 dt}{t+1} = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)dt = 6\int t^2 dt - 6\int t dt + 6\int dt - 6\int \frac{dt}{t+1}$$
$$= 2t^3 - 3t^2 + 6t - 6\ell\eta\left|t+1\right| + c$$

Dado que: $t = \sqrt[6]{x}$

Se tiene: $= 2(\sqrt[6]{x})^3 - 3(\sqrt[6]{x})^2 + 6\sqrt[6]{x} - 6\ell\eta\left|\sqrt[6]{x}+1\right| + c$

Respuesta: $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ell\eta\left|\sqrt[6]{x}+1\right| + c$

9.8.-Encontrar: $\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$

Solución.- Previamente se tiene igual índice por lo cual: $\sqrt{x+1} = t \Rightarrow x = t^2 - 1, dx = 2tdt$, de donde:

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}} = \int \frac{2tdt}{t+t^3} = 2\int \frac{dt}{1+t^2} = 2\operatorname{arc}\tau g t + c$$

Dado que: $t = \sqrt{x+1}$, Se tiene: $= 2\operatorname{arc}\tau g\sqrt{x+1} + c$

Respuesta: $\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}} = 2\operatorname{arc}\tau g\sqrt{x+1} + c$

9.9.-Encontrar: $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx$

Solución.- Previamente se tiene que el m.c.m. de los índices de Las raíces es: 6, por lo cual: $x = t^6 \Rightarrow t = \sqrt[6]{x}, dx = 6t^5 dt$, de donde:

$$\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx = \int \frac{t^3-1}{t^2+1}6t^5 dt = 6\int \frac{t^8-t^5}{t^2+1}dt = 6\int \left(t^6 - t^4 - t^3 + t^2 + t - 1 - \frac{t-1}{t^2+1}\right)dt$$

$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t + c_1 - 3\int \frac{2t-2}{t^2+1}dt$$
$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t + c_1 - 3\int \frac{2t-2}{t^2+1}dt + 6\int \frac{dt}{t^2+1}$$
$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 - \frac{3}{2}t^4 + 2t^3 + 3t^2 - 6t - 3\ell\eta\left|t^2+1\right| + 6\operatorname{arc}\tau g t + c$$

Dado que: $t = \sqrt[6]{x}$, se tiene:

$$= \frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} - 3\ell\eta\left|1+\sqrt[3]{x}\right| + 6\operatorname{arc}\tau g\sqrt[6]{x} + c$$

Respuesta:
$$\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx = \frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} - 3\ell\eta\left|1+\sqrt[3]{x}\right| + 6\operatorname{arc}\tau g\sqrt[6]{x} + c$$

9.10.-Encontrar: $\int \frac{\sqrt{x}dx}{x+2}$

Solución.- La expresión "irracional" es $\sqrt{x}$, por lo tanto:
$\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$,

luego: $\int \frac{\sqrt{x}dx}{x+2} = \int \frac{t(2tdt)}{t^2+2} = 2\int \frac{t^2dt}{t^2+2} = 2\int \left(1 - \frac{2}{t^2+2}\right)dt = 2\int dt - 4\int \frac{dt}{t^2+2}$

$= 2t - \frac{4}{\sqrt{2}}\operatorname{arc}\tau g\frac{t}{\sqrt{2}} + c$, dado que: $t = \sqrt{x}$, se tiene: $= 2\sqrt{x} - 2\sqrt{2}\operatorname{arc}\tau g\sqrt{x/2} + c$

Respuesta: $\int \frac{\sqrt{x}dx}{x+2} = 2\sqrt{x} - 2\sqrt{2}\operatorname{arc}\tau g\sqrt{x/2} + c$

9.11.-Encontrar: $\int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2 - \sqrt{x+1}}$

Solución.- Previamente se tiene igual índice por lo cual: $\sqrt{x+1} = t \Rightarrow x = t^2 - 1, dx = 2tdt$, de donde:

$$\int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2 - \sqrt{x+1}} = \int \frac{\left[(x+1)^{1/2}+2\right]dx}{(x+1)^2 - (x+1)^{1/2}} = \int \frac{t+2}{t^4-t}2tdt = 2\int \frac{(t+2)\cancel{t}dt}{\cancel{t}(t^3-1)}$$

$= 2\int \frac{(t+2)dt}{(t-1)(t^2+t+1)}$ $(*)$, considerando que:

$$\frac{t+2}{(t-1)(t^2+t+1)} = \frac{A}{(t-1)} + \frac{Bt+C}{(t^2+t+1)} \Rightarrow A = 1, B = -1, C = -1$$

Dado que: $t = \sqrt{x+1}$. Se tiene: $= 2\operatorname{arc}\tau g\sqrt{x+1} + c$

$$(*)\ 2\int \frac{(t+2)dt}{(t-1)(t^2+t+1)} = 2\int \frac{dt}{(t-1)} + 2\int \frac{-t-1}{(t^2+t+1)}dt = 2\int \frac{dt}{(t-1)} - 2\int \frac{t+1}{(t^2+t+1)}dt$$

$$= 2\int \frac{dt}{(t-1)} - 2\int \frac{\frac{1}{2}(2t+1)+\frac{1}{2}}{(t^2+t+1)}dt = 2\int \frac{dt}{(t-1)} - \int \frac{(2t+1)dt}{(t^2+t+1)} - \int \frac{dt}{(t^2+t+1)}$$

$$= 2\int \frac{dt}{(t-1)} - \int \frac{(2t+1)dt}{(t^2+t+1)} - \int \frac{dt}{(t^2+t+\frac{1}{4})+\frac{3}{4}}$$

$$= 2\ell\eta\left|t-1\right| - \ell\eta\left|t^2+t+1\right| - \frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2t+1}{\sqrt{3}} + c$$

$$= \ell\eta\left|\frac{(t-1)^2}{(t^2+t+1)}\right| - \frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2t+1}{\sqrt{3}} + c$$

Dado que: $t = \sqrt{x+1}$, se tiene

Respuesta: $\int \frac{(\sqrt{x+1}+2)dx}{(x+1)^2-\sqrt{x+1}} = \ell\eta\left|\frac{(\sqrt{x+1}-1)^2}{(\sqrt{x+1}+x+2)}\right| - \frac{2}{\sqrt{3}}\operatorname{arc}\tau g\,\frac{2\sqrt{x+1}+1}{\sqrt{3}}+c$

EJERCICIOS PROPUESTOS

9.12.- $\int \frac{1+x}{1+\sqrt{x}}dx$ **9.13.-** $\int \frac{1-x}{1+\sqrt{x}}dx$ **9.14.-** $\int \frac{dx}{a+b\sqrt{x}}$

9.15.- $\int \frac{\sqrt{x+a}}{x+a}dx$ **9.16.-** $\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}}$ **9.17.-** $\int \frac{\sqrt{x}-\sqrt[6]{x}}{\sqrt[3]{x}+1}dx$

9.18.- $\int \frac{dx}{x-2-\sqrt{x}}dx$ **9.19.-** $\int \sqrt{\frac{1+x}{1-x}}dx$ **9.20.-** $\int \frac{\sqrt{x+a}}{x+b}dx$

9.21.- $\int \frac{\sqrt[3]{x+1}}{x}dx$ **9.22.-** $\int \frac{\sqrt{a^2-x^2}}{x^3}dx$ **9.23.-** $\int x^2\sqrt{x+a}dx$

9.24.- $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}+2\sqrt[8]{x}}$ **9.25.-** $\int x^3\sqrt{x^2+a^2}dx$

RESPUESTAS

9.12.- $\int \frac{1+x}{1+\sqrt{x}}dx$

Solución.- Sea: $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\int \frac{1+x}{1+\sqrt{x}}dx = \int \frac{1+t^2}{1+t}2tdt = 2\int \frac{t+t^3}{1+t}dt = 2\int\left(t^2-t+2-\frac{2}{t+1}\right)dt$$

$$= 2\int t^2dt - 2\int tdt + 4\int dt - 4\int \frac{dt}{t+1} = \frac{2t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} + 4t - 4\ell\eta\left|t+1\right| + c$$

$$= \frac{2\sqrt{x^3}}{3} - x + 4\sqrt{x} - 4\ell\eta\left|\sqrt{x}+1\right| + c$$

9.13.- $\int \frac{1-x}{1+\sqrt{x}}dx$

Solución.- Sea: $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\int \frac{1-x}{1+\sqrt{x}}dx = \int \frac{1-t}{1+t}2tdt = 2\int \frac{t-t^2}{1+t}dt = -2\int tdt + 4\int dt - 4\int \frac{dt}{t+1} = -t^2 + 4t - 4\ell\eta\left|t+1\right| + c$$

$$= -x + 4\sqrt{x} - 4\ell\eta\left|\sqrt{x}+1\right| + c$$

9.14.- $\int \frac{dx}{a+b\sqrt{x}}$

Solución.- Sea: $\sqrt{x}=t \Rightarrow x=t^2, dx=2tdt$

$$\int \frac{dx}{a+b\sqrt{x}} = \int \frac{2tdt}{a+bt} = 2\int \frac{tdt}{a+bt} = 2\int \left(\frac{1}{b} - \frac{a}{b}\frac{1}{a+bt}\right)dt = \frac{2}{b}\int dt - \frac{2a}{b^2}\int \frac{bdt}{a+bt}$$

$$= \frac{2}{b}t - \frac{2a}{b^2}\ell\eta\left|a+bt\right| + c = \frac{2}{b}\sqrt{x} - \frac{2a}{b^2}\ell\eta\left|a+b\sqrt{x}\right| + c$$

9.15.- $\int \frac{\sqrt{x+a}}{x+a}dx$

Solución.- Sea: $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\int \frac{\sqrt{x+a}}{x+a}dx = \int \frac{\cancel{t}\,2\cancel{t}\,dt}{\cancel{t^2}} = 2\int dt = 2t + c = 2\sqrt{x+a} + c$$

9.16.- $\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}}$

Solución.- m.c.m: 4 ; Sea: $\sqrt[4]{x} = t \Rightarrow x = t^4, dx = 4t^3dt$

$$\int \frac{\sqrt{x}dx}{1+\sqrt[4]{x}} = \int \frac{t^2\,4t^3dt}{1+t} = 4\int \frac{t^5dt}{1+t} = 4\int \left(t^4 - t^3 + t^2 - t + 1 - \frac{1}{t+1}\right)dt$$

$$= 4\left(\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + t - \ell\eta\left|t+1\right|\right) + c = \frac{4t^5}{5} - t^4 + \frac{4t^3}{3} - 2t^2 + 4t - 4\ell\eta\left|t+1\right|$$

$$= \frac{4x^{5/4}}{5} - x + \frac{4x^{3/4}}{3} - 2x^{1/2} + 4x^{1/4} - 4\ell\eta\left|x^{1/4} + 1\right|$$

9.17.- $\int \frac{\sqrt{x} - \sqrt[6]{x}}{\sqrt[3]{x} + 1}dx$

Solución.- m.c.m: 6 ; Sea: $\sqrt[6]{x} = t \Rightarrow x = t^6, dx = 6t^5dt$

$$\int \frac{\sqrt{x} - \sqrt[6]{x}}{\sqrt[3]{x} + 1}dx = \int \frac{t^3 - t}{t^2 + 1}6t^5dt = 6\int \frac{(t^8 - t^6)dt}{t^2 + 1} = 6\int t^6dt - 2\int t^4dt + 2\int t^2dt - 2\int dt + 2\int \frac{dt}{1+t^2}$$

$$= 6\left(\frac{t^7}{7} - \frac{2t^5}{5} + \frac{2t^3}{3} - 2t + 2\operatorname{arc}\tau gt\right) + c = \frac{6t^7}{7} - \frac{12t^5}{5} + 4t^3 - 12t + 12\operatorname{arc}\tau gt + c$$

$$= \frac{6x^{7/6}}{7} - \frac{12x^{5/2}}{5} + 4x^{1/2} - 12x^{1/6} + 12\operatorname{arc}\tau g x^{1/6} + c$$

9.18.- $\int \frac{dx}{x - 2 - \sqrt{x}}dx$

Solución.- Sea: $\sqrt{x} = t \Rightarrow x = t^2, dx = 2tdt$

$$\int \frac{dx}{x - 2 - \sqrt{x}}dx = \int \frac{2tdt}{t^2 - 2 - t} = \int \frac{(2t-1)+1}{t^2 - t - 2}dt = \int \frac{2t-1}{t^2 - t - 2}dt + \int \frac{dt}{t^2 - t - 2}$$

$$= \int \frac{2t-1}{t^2 - t - 2}dt + \int \frac{dt}{(t - 1/2)^2 - 9/4} = \ell\eta\left|t^2 - t - 2\right| + \frac{1}{\cancel{2}\,\frac{3}{\cancel{2}}}\ell\eta\left|\frac{t - 3/2}{t + 3/2}\right| + c$$

$$= \ell\eta\left|t^2 - t - 2\right| + \frac{1}{3}\ell\eta\left|\frac{2t-3}{2t+3}\right| + c = \ell\eta\left|x - \sqrt{x} - 2\right| + \frac{1}{3}\ell\eta\left|\frac{2\sqrt{x}-3}{2\sqrt{x}+3}\right| + c$$

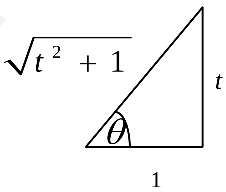
9.19.- $\int\sqrt{\frac{1+x}{1-x}}dx$

Solución.- Notará el lector, que este caso se diferencia de los anteriores, sin embargo la técnica que se seguirá, tiene la misma fundamentación y la información que se consiga es valiosa. $(*)$

Sea: $\sqrt{\frac{1+x}{1-x}} = t \Rightarrow \frac{1+x}{1-x} = t^2 \Rightarrow 1 + x = t^2 - t^2x \Rightarrow x(1+t^2) = t^2 - 1$

$x = \frac{t^2-1}{t^2+1} \Rightarrow dx = \frac{4tdt}{(t^2+1)^2}$, luego:

$(*)$ $\int\sqrt{\frac{1+x}{1-x}}dx = \int\frac{t4tdt}{(t^2+1)^2} = \int\frac{4t^2dt}{(t^2+1)^2} = 4\int\frac{t^2dt}{(\sqrt{t^2+1})^4}(**)$, haciendo uso de sustituciones trigonométricas convenientes en $(**)$, y de la figura se tiene:



Se tiene: $t = \tau g\theta, dt = \sec^2\theta d\theta; \sqrt{t^2+1} = \sec\theta$

$$(**)\ 4\int\frac{t^2dt}{(\sqrt{t^2+1})^4} = \int\frac{4\tau g^2\theta\cancel{\sec^2\theta}d\theta}{\sec^{\cancel{4}}\theta} = 4\int\frac{\tau g^2\theta}{\sec^2\theta}d\theta$$

$$= 4\int \mathrm{s\,e\,n}^2\theta d\theta = 2\int d\theta - 2\int\cos 2\theta d\theta = 2\theta - \mathrm{s\,e\,n}\,2\theta + c = 2\theta - 2\,\mathrm{s\,e\,n}\,\theta\cos\theta + c$$

$$= 2\,\mathrm{arc}\,\tau g t - 2\frac{t}{\sqrt{t^2+1}}\frac{1}{\sqrt{t^2+1}} + c = 2\,\mathrm{arc}\,\tau g t - \frac{2t}{t^2+1} + c = 2\,\mathrm{arc}\,\tau g\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\frac{1+x}{1-x}}}{\frac{1+x}{1-x}+1} + c$$

$$= 2\,\mathrm{arc}\,\tau g\sqrt{\frac{1+x}{1-x}} - (1-x)\sqrt{\frac{1+x}{1-x}} + c$$

9.20.- $\int\frac{\sqrt{x+a}}{x+b}dx$

Solución.- Sea: $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\int\frac{\sqrt{x+a}}{x+b}dx = \int\frac{t2tdt}{t^2-a+b} = 2\int\frac{t^2dt}{t^2+(b-a)} = 2\int\left(1 - \frac{b-a}{t^2+(b-a)}\right)dt$$

$$= 2\int dt - 2(b-a)\int\frac{dt}{t^2+(b-a)} = 2t - 2(b-a)\frac{1}{\sqrt{b-a}}\,\mathrm{arc}\,\tau g\,\frac{t}{\sqrt{b-a}} + c$$

$= 2\sqrt{x+a} - 2\sqrt{b-a}\,\mathrm{arc}\,\tau g\sqrt{\frac{x+a}{b-a}} + c$

9.21.- $\int \frac{\sqrt[3]{x+1}}{x}dx$

Solución.- Sea: $\sqrt[3]{x+1} = t \Rightarrow x = t^3 - 1, dx = 3t^2dt$

$$\int \frac{\sqrt[3]{x+1}}{x}dx = \int \frac{t3t^2dt}{t^3-1} = 3\int \frac{t^3dt}{t^3-1} = 3\int \left(1 + \frac{1}{t^3-1}\right)dt = 3\int dt + 3\int \frac{dt}{t^3-1}$$

$= 3\int dt + 3\int \frac{dt}{(t-1)(t^2+t+1)}(*)$, por fracciones parciales:

$\frac{3}{(t-1)(t^2+t+1)} = \frac{A}{(t-1)} + \frac{Bt+C}{(t^2+t+1)} \Rightarrow 3 = A(t^2+t+1) + (Bt+C)(t-1)$, de donde:

$A = 1, B = -1, C = -2$, luego:

$$(*) = 3\int dt + \int \frac{dt}{t-1} - \int \frac{t+2}{t^2+t+1}dt = 3t + \ell\eta|t-1| - \tfrac{1}{2}\ell\eta|t^2+t+1| - \sqrt{3}\,\mathrm{arc}\,\tau g\left(\frac{2t+1}{\sqrt{3}}\right) + c$$

9.22.- $\int \frac{\sqrt{a^2-x^2}}{x^3}dx$

Solución.- Sea: $\sqrt{a^2-x^2} = t \Rightarrow x^2 = a^2 - t^2, xdx = -tdt$

$$\int \frac{\sqrt{a^2-x^2}}{x^3}dx = \int \frac{\sqrt{a^2-x^2}\,xdx}{x^4} = -\int \frac{ttdt}{(a^2-t^2)^2} = \int \frac{-t^2dt}{(a^2-t^2)^2} = \int \frac{-t^2dt}{(a+t)^2(a-t)^2}(*)$$

Por fracciones parciales:

$\frac{-t^2}{(t+a)^2(t-a)^2} = \frac{A}{(t+a)} + \frac{B}{(t+a)^2} + \frac{C}{(t-a)} + \frac{D}{(t-a)^2}$, de donde:

$A = \tfrac{1}{4}a, B = -\tfrac{1}{4}, C = -\tfrac{1}{4}a, D = -\tfrac{1}{4}$, luego:

$$(*)\int \frac{-t^2dt}{(a+t)^2(a-t)^2} = \frac{1}{4a}\int \frac{dt}{(t+a)} - \frac{1}{4a}\int \frac{dt}{(t+a)^2} - \frac{1}{4a}\int \frac{dt}{(t-a)} - \frac{1}{4a}\int \frac{dt}{(t-a)^2}$$

$$= \frac{1}{4a}\ell\eta|(t+a)| + \frac{1}{4(t+a)} - \frac{1}{4a}\ell\eta|(t-a)| + \frac{1}{4(t-a)} + c$$

$$= \frac{1}{4a}\ell\eta\left|\frac{(t+a)}{(t-a)}\right| + \frac{1}{4(t+a)} + \frac{1}{4(t-a)} + c$$

$$= \frac{1}{4a}\ell\eta\left|\frac{\sqrt{a^2-x^2}+a}{\sqrt{a^2-x^2}-a}\right| + \frac{\sqrt{a^2-x^2}}{2(\cancel{a^2}-x^2-\cancel{a^2})} + c = \frac{1}{4a}\ell\eta\left|\frac{\sqrt{a^2-x^2}+a}{\sqrt{a^2-x^2}-a}\right| - \frac{\sqrt{a^2-x^2}}{2x^2} + c$$

$$= \frac{1}{4a}\ell\eta\left|\frac{(\sqrt{a^2-x^2}+a)^2}{\cancel{a^2}-x^2-\cancel{a^2}}\right| - \frac{\sqrt{a^2-x^2}}{2x^2} + c = \frac{1}{2a}\ell\eta\left|\sqrt{a^2-x^2}+a\right| - \frac{1}{2a}\ell\eta|x| - \frac{\sqrt{a^2-x^2}}{2x^2} + c$$

9.23.- $\int x^2\sqrt{x+a}\,dx$

Solución.- Sea: $\sqrt{x+a} = t \Rightarrow x = t^2 - a, dx = 2tdt$

$$\int x^2\sqrt{x+a}\,dx=\int (t^2-a)^2 t2tdt=2\int t^2(t^2-a)^2dt=2\int (t^6-2at^4+a^2t^2)dt$$
$$=2\int t^6dt-4a\int t^4dt+2a^2\int t^2dt=\frac{2t^7}{7}-\frac{4at^5}{5}+\frac{2a^2t^3}{3}+c$$
$$=\frac{2(x+a)^{7/2}}{7}-\frac{4a(x+a)^{5/2}}{5}+\frac{2a^2(x+a)^{3/2}}{3}+c$$

9.24.- $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}+2\sqrt[8]{x}}$

Solución.- Sea: $\sqrt[8]{x}=t \Rightarrow x=t^8, dx=8t^7dt$

$$\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}+2\sqrt[8]{x}}=\int \frac{8t^7dt}{t^4+t^2+2t}=8\int \frac{t^6dt}{t^3+t+2}=8\int\left(t^3-t-2+\frac{t^2+4t+4}{t^3+t+2}\right)dt$$
$$=8\int t^3-8\int tdt-16\int dt+8\int \frac{t^2+4t+4}{t^3+t+2}dt=8\frac{t^4}{4}-\frac{8t^2}{2}-16t+8\int \frac{t^2+4t+4}{t^3+t+2}dt$$
$=2t^4-4t^2-16t+8\int \frac{t^2+4t+4}{t^3+t+2}dt(*)$, por fracciones parciales:

$\frac{t^2+4t+4}{(t^3+t+2)}=\frac{t^2+4t+4}{(t+1)(t^2-t+2)}=\frac{A}{(t+1)}+\frac{Bt+C}{(t^2-t+2)}\Rightarrow A=\frac{1}{4}, B=\frac{3}{4}, C=\frac{14}{4}$, luego:

$$(*)=2t^4-4t^2-16t+8\left(\int \frac{\frac{1}{4}dt}{t+1}+\int \frac{\frac{3}{4}t+\frac{14}{4}}{t^2-t+2}dt\right)$$
$$=2t^4-4t^2-16t+8\left(\frac{1}{4}\int \frac{dt}{t+1}+\frac{1}{4}\int \frac{3t+14}{t^2-t+2}dt\right)=2t^4-4t^2-16t+2\int \frac{dt}{t+1}+2\int \frac{3t+14}{t^2-t+2}dt$$
$$=2t^4-4t^2-16t+2\ell\eta|t+1|+\cancel{2}\frac{3}{\cancel{2}}\int \frac{2t+\frac{28}{3}-\frac{31}{3}+\frac{31}{3}}{t^2-t+2}dt$$
$$=2t^4-4t^2-16t+2\ell\eta|t+1|+3\int \frac{(2t-1)}{t^2-t+2}dt+31\int \frac{dt}{t^2-t+2}$$
$$=2t^4-4t^2-16t+2\ell\eta|t+1|+3\ell\eta|t^2-t+2|+31\int \frac{dt}{(t-\frac{1}{2})^2+\frac{7}{4}}$$
$$=2t^4-4t^2-16t+2\ell\eta|t+1|+3\ell\eta|t^2-t+2|+31\frac{2}{\sqrt{7}}\operatorname{arc}\tau g\,\frac{t-\frac{1}{2}}{\frac{\sqrt{7}}{2}}+c$$
$$=2t^4-4t^2-16t+2\ell\eta|t+1|+3\ell\eta|t^2-t+2|+\frac{62}{\sqrt{7}}\operatorname{arc}\tau g\,\frac{2t-1}{\sqrt{7}}+c$$
$$=2x^{1/2}-4x^{1/4}-16x^{1/8}+2\ell\eta\left|x^{1/8}+1\right|+3\ell\eta\left|x^{1/4}-x^{1/8}+2\right|+\frac{62}{\sqrt{7}}\operatorname{arc}\tau g\,\frac{2x^{1/8}-1}{\sqrt{7}}+c$$

9.25.- $\int x^3\sqrt{x^2+a^2}\,dx$

Solución.- Sea: $\sqrt{x^2+a^2}=t \Rightarrow x^2=t^2-a^2, xdx=tdt$

$$\int x^3\sqrt{x^2+a^2}\,dx=\int x^2\sqrt{x^2+a^2}\,xdx=\int (t^2-a^2)ttdt=\int (t^2-a^2)t^2dt=\int (t^4-a^2t^2)dt$$
$$=\frac{t^5}{5}-\frac{a^2t^3}{3}+c=\frac{(x^2+a^2)^{5/2}}{5}-\frac{a^2(x^2+a^2)^{3/2}}{3}+c=(x^2+a^2)^{3/2}\left(\frac{x^2+a^2}{5}-\frac{a^2}{3}\right)+c$$
$$=(x^2+a^2)^{3/2}\left(\frac{3x^2-2a^2}{15}\right)+c$$