

Integración por sustitución o cambio de variable

$$a) \int \sqrt[3]{x+5} dx = \int (x+5)^{\frac{1}{3}} dx = \frac{(x+5)^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$b) \int \sqrt[5]{3+4x} dx = \int (3+4x)^{\frac{1}{5}} dx = \frac{1}{4} \frac{(3+4x)^{\frac{6}{5}}}{\frac{6}{5}} + C$$

$$c) \int \sec^2 5x dx = \frac{1}{5} \operatorname{tg} 5x + C$$

$$d) \int \cos 2x dx = \frac{1}{2} \operatorname{sen} 2x + C$$

$$e) \int \frac{\operatorname{sen} 2x}{1+\cos^2 x} dx = - \int \frac{\operatorname{sen} 2x}{t} \frac{dt}{\operatorname{sen} 2x} = - \int \frac{dt}{t} = -\ln t = -\ln(1+\cos^2 x) + C$$

$$\begin{cases} t = 1 + \cos^2 x \\ dt = -2 \cos x \operatorname{sen} x dx \end{cases}$$

$$f) \int \frac{e^{2x}}{1+e^x} dx = \int \frac{(t-1)^2}{t} \frac{dt}{t-1} = \int \frac{t-1}{t} dt = \int dt - \int \frac{1}{t} dt = t - \ln t = 1 + e^x - \ln(1+e^x) + C$$

$$\begin{cases} t = 1 + e^x; \\ dt = e^x dx; \quad dx = \frac{dt}{t-1} \end{cases}$$

$$g) \int \frac{2 \cos x}{4 + \operatorname{sen} x} dx = 2 \int \frac{\operatorname{cosec} t}{t} \frac{dt}{\operatorname{cosec} t} = 2 \int \frac{dt}{t} = 2 \ln t = 2 \ln(4 + \operatorname{sen} x) + C$$

$$\begin{cases} t = 4 + \operatorname{sen} x \\ dt = \cos x dx \end{cases}$$

$$h) \int \operatorname{sen}^2 \frac{x}{2} \cos \frac{x}{2} dx = 2 \int t^2 \frac{dt}{\operatorname{cosec} \frac{x}{2}} = \frac{2t^3}{3} = \frac{2}{3} \operatorname{sen}^3 \frac{x}{2} + C$$

$$\begin{cases} t = \operatorname{sen} \frac{x}{2} \\ dt = \frac{1}{2} \cos \frac{x}{2} dx; \quad dx = \frac{2dt}{\cos \frac{x}{2}} \end{cases}$$

$$i) \int (1 + \operatorname{tg}^2 x) dx = \int (1 + \operatorname{tg}^2 x + 2 \operatorname{tg} x) dx = \int \left(\sec^2 x + 2 \frac{\operatorname{sen} x}{\cos x} \right) dx = \operatorname{tg} x + 2 \ln x + C$$

$$j) \int \frac{\sec^2 x}{a + b \operatorname{tg} x} dx = \frac{1}{b} \int \frac{b \sec^2 x}{a + b \operatorname{tg} x} dx = \frac{1}{b} \ln(a + b \operatorname{tg} x) + C$$

$$k) \int \frac{\operatorname{sen} 3x}{\sqrt{5 + \cos 3x}} dx = -\frac{1}{3} \int \frac{\operatorname{sen} 3x}{t^{\frac{1}{2}}} \frac{dt}{\operatorname{sen} 3x} = -\frac{1}{3} \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{2}{3} \sqrt{5 + \cos x} + C$$

$$\begin{cases} 5 + \cos 3x = t; \\ -3 \operatorname{sen} 3x dx = dt; \quad dx = \frac{dt}{-3 \operatorname{sen} 3x} \end{cases}$$

$$l) \int \frac{x^2}{(3+2x^3)^2} dx = \frac{1}{6} \int \frac{x^2}{t^2} \frac{dt}{x^2} = \frac{1}{6} \int t^{-2} dt = -\frac{1}{6t} = -\frac{1}{6(3+2x^3)^2} + C$$

$$\begin{cases} 3+2x^3 = t; \\ 6x^2 dx = dt; \quad dx = \frac{dt}{6x^2} \end{cases}$$

$$m) \int \frac{ax^m}{1+bx^{n+1}} dx = \frac{a}{(n+1)b} \int \frac{x^m}{t} \frac{dt}{x^m} = \frac{a}{(n+1)b} \int \frac{dt}{t} =$$

$$= \frac{a}{(n+1)b} \ln t = \frac{a}{(n+1)b} \ln(1+bx^{n+1}) + C$$

$$\begin{cases} 1+bx^{n+1} = t; \\ (n+1)bx^n dx = dt; \quad dx = \frac{dt}{(n+1)bx^n} \end{cases}$$

$$n) \int \frac{e^x}{4-3e^x} dx = \frac{1}{3} \int \frac{e^x}{t} \frac{dt}{e^x} = \frac{1}{3} \ln t = \frac{1}{3} \ln(4-3e^x) + C$$

$$\begin{cases} 4-3e^x; \\ -3e^x dx = dt; \quad dx = \frac{dx}{-3e^x} \end{cases}$$

Integración por partes

Sean las funciones $u = f(x)$ y $v = g(x)$; $d(u \cdot v) = u dv + v du \Rightarrow u dv = d(u \cdot v) - v du \Rightarrow \int u dv = \int d(uv) - \int v du$

Se hacen dos partes del integrando, una de ellas se iguala a u y la otra junto con dx a dv , la parte igualada a dv debe ser fácilmente integrable y la nueva integral que aparece tiene que ser igual o menos complicada que la dada al principio.

$$a) \int x^2 \cos x dx = x^2 \operatorname{sen} x - 2 \int x \operatorname{sen} x dx = x^2 \operatorname{sen} x - 2(-x \cos x - \int \cos x dx) =$$

$$= x^2 \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x + C$$

$u = x^2 \quad du = 2x dx$	$u = x \quad du = dx$
$dv = \cos x dx \quad v = \int \cos x dx = \operatorname{sen} x$	$dv = \operatorname{sen} x dx \quad v = \int \operatorname{sen} x dx = -\cos x$

$$b) \int \ln x dx = x \ln x - \int dx = x \ln x - x + C$$

$u = \ln x \quad du = \frac{1}{x} dx$
$dv = dx \quad v = \int dx = x$

$$c) \int x \ln x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$\boxed{\begin{array}{l} u = \ln x \quad du = \frac{1}{x} dx \\ dv = x dx \quad v = \int x dx = \frac{x^2}{2} \end{array}}$$

$$d) \int x^2 e^{2x} dx = \frac{x^2}{2} e^{2x} - \int x e^{2x} dx = \frac{x^2}{2} e^{2x} - \left(\frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx \right) =$$

$$= \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} = e^{2x} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) + C$$

$$\boxed{\begin{array}{l} u = x^2 \quad du = 2x dx \\ dv = e^{2x} dx \quad v = \frac{1}{2} e^{2x} \end{array}} \quad \boxed{\begin{array}{l} u = x \quad du = dx \\ dv = e^{2x} dx \quad v = \frac{1}{2} e^{2x} \end{array}}$$

$$e) \int x \sqrt{1+x} dx = 2 \int (t^2 - 1) \cdot t \cdot t dt = 2 \int (t^4 - t^2) dt =$$

$$= 2 \left(\frac{t^5}{5} - \frac{t^3}{3} \right) = \frac{2(\sqrt{1+x})^5}{5} - \frac{2(\sqrt{1+x})^3}{3} + C$$

$$\sqrt{1+x} = t; \quad x = t^2 - 1$$

$$\frac{1}{2\sqrt{1+x}} dx = dt \quad dx = 2t dt$$

$$f) \int x \sec^2 3x dx = \frac{x}{3} \operatorname{tg} 3x - \frac{1}{3} \int \operatorname{tg} 3x dx = \frac{x}{3} \operatorname{tg} 3x + \frac{1}{9} \ln \cos 3x + C$$

$$\boxed{\begin{array}{l} u = x \quad du = dx \\ dv = \sec^2 3x \quad v = \frac{1}{3} \operatorname{tg} 3x \end{array}} \int \operatorname{tg} 3x dx = \int \frac{\operatorname{sen} 3x}{\cos 3x} dx = -\frac{1}{3} \ln \cos x$$

$$g) \int \operatorname{sen} 3x \operatorname{sen} x dx = -\operatorname{sen} 3x \cos x - 3 \int \cos 3x \cos x dx = -\operatorname{sen} 3x \cos x +$$

$$+ 3 \left(\cos 3x \operatorname{sen} x - 3 \int \operatorname{sen} 3x \operatorname{sen} x dx \right) = -\operatorname{sen} 3x \cos x + 3 \cos 3x \operatorname{sen} x - 9 \int \operatorname{sen} 3x \operatorname{sen} x dx$$

$$\boxed{\begin{array}{l} u = \operatorname{sen} 3x \quad du = 3 \cos x dx \\ dv = \operatorname{sen} x dx \quad v = -\cos x \end{array}} \quad \boxed{\begin{array}{l} u = \cos 3x \quad du = -3 \operatorname{sen} 3x dx \\ dv = \cos x dx \quad v = \operatorname{sen} x \end{array}}$$

$$\text{Llamamos } I = \int \operatorname{sen} 3x \operatorname{sen} x dx \Rightarrow \begin{cases} I = -\operatorname{sen} 3x \cos x + 3 \cos 3x \operatorname{sen} x + 9I \Rightarrow \\ \Rightarrow I = \frac{\operatorname{sen} 3x \cos x - 3 \cos 3x \operatorname{sen} x}{8} \end{cases}$$

$$h) \int x \operatorname{arctg} x dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} (x - \operatorname{arctg} x) + C$$

$$\boxed{\begin{array}{l} u = \operatorname{arctg} x \quad du = \frac{1}{1+x^2} dx \\ dv = x dx \quad v = \frac{x^2}{2} \end{array}} \int \frac{x^2}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2} \right) dx = x - \operatorname{arctg} x$$

$$\frac{x^2}{-x^2-1} \quad \frac{1}{1+x^2} \quad D = d \times c + r \Rightarrow \quad \frac{D}{d} = \frac{\cancel{d} \times c}{\cancel{d}} + \frac{r}{d}$$

$$-1$$

$$I = \int e^{nx} \operatorname{sen} bx \, dx = -\frac{\cos bx e^{nx}}{b} + \frac{n}{b} \int e^{nx} \cos bx \, dx = -\frac{\cos bx e^{nx}}{b} +$$

$$i) \quad + \frac{n}{b} \left(\frac{\operatorname{sen} bx e^{nx}}{b} - \frac{n}{b} \int e^{nx} \operatorname{sen} bx \, dx \right) = -\frac{\cos bx e^{nx}}{b} + \frac{n \operatorname{sen} bx e^{nx}}{b} - \frac{n^2}{b^2} I \Rightarrow$$

$$\left(1 + \frac{n^2}{b^2} \right) I = \frac{e^{nx}}{b} \left(\frac{n}{b} \operatorname{sen} x - \cos bx \right) \Rightarrow \frac{b^2 + n^2}{b^2} I = \frac{e^{nx}}{b} \left(\frac{n}{b} \operatorname{sen} x - \cos bx \right) \Rightarrow$$

$$\Rightarrow I = \frac{\cancel{b} \left(\frac{n}{b} \operatorname{sen} x - \cos bx \right) b^{\cancel{2}}}{b^2 + n^2} = \frac{b e^{nx}}{b^2 + n^2} \left(\frac{n}{b} \operatorname{sen} x - \cos bx \right) + C$$

$$j) \quad \int \frac{1+x}{1+x^2} dx = \int \frac{dx}{1+x^2} + \frac{1}{2} \int \frac{2x}{1+x^2} dx = \operatorname{arctg} x + \frac{1}{2} \ln(1+x^2) + C$$

$$k) \quad \int x e^{4x} dx = \frac{x}{4} e^{4x} - \frac{1}{4} \int e^{4x} dx = \frac{x}{4} e^{4x} - \frac{1}{16} e^{4x} + C$$

$u = x \quad du = dx$
$dv = e^{4x} dx \quad v = \frac{1}{4} e^{4x}$

$$l) \quad \int \frac{x}{x^2+9} dx = \frac{1}{2} \int \frac{2x}{x^2+9} dx = \frac{1}{2} \ln(x^2+9) + C$$

$$m) \quad \int e^{-x} \cos x \, dx = -\cos x e^{-x} - \int e^{-x} \operatorname{sen} x \, dx = -\cos x e^{-x} + \operatorname{sen} x e^{-x} - \int e^{-x} \cos x \, dx$$

$u = \cos x \quad du = -\operatorname{sen} x \, dx$
$dv = e^{-x} dx \quad v = -e^{-x}$

$u = \operatorname{sen} x \quad du = \cos x \, dx$
$dv = e^{-x} dx \quad v = -e^{-x}$

Llamamos $I = \int e^{-x} \operatorname{sen} x \, dx \Rightarrow$

$$I = \operatorname{sen} x e^{-x} - \cos x e^{-x} - I \Rightarrow I = \frac{\operatorname{sen} x e^{-x} - \cos x e^{-x}}{2} = \frac{e^{-x}}{2} (\operatorname{sen} x - \cos x) + C$$

$$n) \quad \int x^3 e^{-x^2} dx = \frac{1}{2} \int x^3 e^{-t} \frac{dt}{x} = \frac{1}{2} \int t e^{-t} dt = \frac{1}{2} \left(-t e^{-t} - \int e^{-t} dt \right) = \frac{1}{2} \left(-x^2 e^{-x^2} - e^{-x^2} \right) + C$$

$x^2 = t \quad u = t \quad du = dt$
$2x dx = dt \quad dv = e^{-t} dx \quad v = -e^{-t}$

$$o) \quad \int \frac{1+x}{1+\sqrt{x}} dx = 2 \int \frac{1+t^2}{1+t} \cdot t \cdot dt = 2 \int \frac{t^3+t}{t+1} dt = 2 \int \left(t^2 - t + 2 - \frac{2}{t+1} \right) dt =$$

$$2 \left(\frac{t^3}{3} - \frac{t^2}{2} + 2t - \ln(t+1) \right) = \frac{2(\sqrt{x})^3}{3} - \frac{(\sqrt{x})^2}{2} + 2\sqrt{x} - \ln|\sqrt{x}+1| + C$$

$\sqrt{x} = t; x = t^2$	$-1 \left \begin{array}{cccc} 1 & 0 & 1 & 0 \\ & -1 & 1 & -2 \\ \hline 1 & -1 & 2 & -2 \end{array} \right $
$dx = 2t dt$	

$$p) \quad \int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx = -x^2 e^{-x} + 2 \left(-x e^{-x} + \int e^{-x} dx \right) = -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) =$$

$$= -e^{-x} (x^2 + 2x + 2) + C$$

$u = x^2 \quad du = 2x dx$
$dv = e^{-x} dx \quad v = -e^{-x} dx$

$u = x \quad du = dx$
$dv = e^{-x} dx \quad v = -e^{-x}$

$$\begin{aligned}
 \text{q)} \quad \int (x^2 + 1)e^{-2x} dx &= -\frac{(x^2 + 1)e^{-2x}}{2} + \frac{2}{2} \int x e^{-2x} dx = \frac{(x^2 + 1)e^{-2x}}{2} - \frac{1}{2} x e^{-2x} + \\
 &+ \frac{1}{2} \int e^{-2x} dx = \frac{(x^2 + 1)e^{-2x}}{2} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C
 \end{aligned}$$

$u = x^2 + 1 \quad du = 2x dx$	$u = x \quad du = dx$
$dv = e^{-2x} dx \quad v = -\frac{1}{2} e^{-2x}$	$dv = e^{-2x} dx \quad v = -\frac{1}{2} e^{-2x}$

$$\text{r)} \quad \int \frac{dx}{\sqrt{x} \cos^2 \sqrt{x}} = 2 \int \frac{\cancel{t} dt}{\cancel{t} \cos^2 t} = 2 \int \frac{dt}{\cos^2 t} = 2 \int \sec^2 t dt = 2 \operatorname{tg} t = 2 \operatorname{tg} \sqrt{x} + C$$

$$dt = \frac{1}{2\sqrt{x}} dx; \quad dx = 2\sqrt{x} dx$$

$$\begin{aligned}
 \text{s)} \quad \int x [\operatorname{sen} 2x + \ln(1 + x^2)] dx &= I_1 + I_2 = \\
 &= -\frac{1}{2} x \cos 2x + \frac{1}{4} \operatorname{sen} 2x + \frac{1}{2} (x^2 + 1) \ln(x^2 + 1) - \frac{1}{2} (x^2 + 1) + C
 \end{aligned}$$

$$I_1 = \int x \operatorname{sen} 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \operatorname{sen} 2x$$

$u = x \quad du = dx$
$dv = \operatorname{sen} 2x dx \quad v = -\frac{1}{2} \cos 2x$

$$I_2 = \int x \ln(x^2 + 1) dx = \frac{1}{2} \int \cancel{x} \ln t \frac{dt}{\cancel{x}} = \frac{1}{2} \int \ln t dt = \frac{1}{2} (t \ln t - t) =$$

$$= \frac{1}{2} [(x^2 + 1) \ln(x^2 + 1) - (x^2 + 1)]$$

$t = 1 + x^2$	$u = \ln t \quad du = \frac{1}{t} dt$
$dt = 2x dx$	$dv = dt \quad v = t$