Problema 1 Calcular los siguientes límites:

a) $\lim_{x\longrightarrow 0}\dfrac{\ln(\cos 2x)}{x^2}$ (Castilla León 2006)

b) Determinar los valores a y b para los que

$$\lim_{x\longrightarrow 0}\frac{ax^2+bx+1-\cos x}{\sin x^2}=1$$

(Castilla León 2006)

c) $\lim_{x\longrightarrow 0}\dfrac{1+x-e^x}{\sin^2 x}$ (Extremadura 2006)

d) Si $f(x)\frac{(x+1)^2}{e^x}$ calcular $\lim_{x\longrightarrow\infty} f(x)$ y $\lim_{x\longrightarrow -\infty} f(x)$.

(Islas Baleares 2006)

e) Calcular si existen los siguientes límites:

- $\lim_{x\longrightarrow 0}(\sin x)^{\tan x}$
- $\lim_{x\longrightarrow 0}\dfrac{\sin x}{|x|}$
- $\lim_{x\longrightarrow a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$ con $(x>0)$

(La Rioja 2006)

f) $\lim_{x\longrightarrow 0}\dfrac{\sin x}{7x^2}$ (Zaragoza)

g) Si $a_n=\frac{2n}{n+1}$ calcular $\lim_{x\longrightarrow\infty} n^2(a_{n+1}-a_n)$ (Madrid 2006)

Solución:

a) $\lim_{x\longrightarrow 0}\dfrac{\ln(\cos 2x)}{x^2}=\left[\dfrac{0}{0}\right]=\lim_{x\longrightarrow 0}\dfrac{\frac{-2\sin 2x}{\cos 2x}}{2x}=\lim_{x\longrightarrow 0}\dfrac{-\tan 2x}{x}=\left[\dfrac{0}{0}\right]=$
$\lim_{x\longrightarrow 0}\dfrac{-2}{\cos^2 x}=-2$

b) $\lim_{x\longrightarrow 0}\dfrac{ax^2+bx+1-\cos x}{\sin x^2}=\left[\dfrac{0}{0}\right]=\lim_{x\longrightarrow 0}\dfrac{2ax+b+\sin x}{2x\cos x^2}=\left[\dfrac{0}{0}\right]=$
(con $b=0$) $=\lim_{x\longrightarrow 0}\dfrac{2a+\cos x}{2\cos x^2-4x^2\sin x^2}=\dfrac{2a+1}{2}=1\Longrightarrow a=\dfrac{1}{2}$

c) $\lim_{x\longrightarrow 0}\dfrac{1+x-e^x}{\sin^2 x}=\left[\dfrac{0}{0}\right]=\lim_{x\longrightarrow 0}\dfrac{1-e^x}{2\cos x\sin x}=\lim_{x\longrightarrow 0}\dfrac{1-e^x}{\sin 2x}=\left[\dfrac{0}{0}\right]=$
$\lim_{x\longrightarrow 0}\dfrac{-e^x}{2\cos 2x}=-\dfrac{1}{2}$

d)
- $\lim_{x\longrightarrow\infty}\dfrac{(x+1)^2}{e^x}=\left[\dfrac{\infty}{\infty}\right]=\lim_{x\longrightarrow\infty}\dfrac{2(x+1)}{e^x}=\left[\dfrac{\infty}{\infty}\right]=\lim_{x\longrightarrow\infty}\dfrac{2}{e^x}=0$
- $\lim_{x\longrightarrow-\infty}\dfrac{(x+1)^2}{e^x}=[\infty\cdot\infty]=\infty$

e)
- $\lim_{x\longrightarrow 0}(\sin x)^{\tan x}=\lambda\Longrightarrow\lim_{x\longrightarrow 0}\tan x\ln\sin x=\ln\lambda$

  $\lim_{x\longrightarrow 0}\tan x\ln\sin x=\lim_{x\longrightarrow 0}\dfrac{\ln\sin x}{\frac{1}{\tan x}}=\left[\dfrac{-\infty}{\infty}\right]=\lim_{x\longrightarrow 0}\dfrac{\frac{\cos}{\sin x}}{\frac{-1/\cos^2 x}{\tan^2 x}}=$
  $\lim_{x\longrightarrow 0}(-\cos x\tan x)=0\Longrightarrow\ln\lambda=0\Longrightarrow\lambda=1$
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$$\frac{\sin x}{|x|}=\begin{cases}-\frac{\sin x}{x} & \text{si}\quad x<0\\ \frac{\sin x}{x} & \text{si}\quad x\geq 0\end{cases}$$
$$\lim_{x\longrightarrow 0^-}\frac{\sin x}{x}=\lim_{x\longrightarrow 0}-\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\longrightarrow 0}-\frac{\cos x}{1}=-1$$
$$\lim_{x\longrightarrow 0^+}\frac{\sin x}{x}=\lim_{x\longrightarrow 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]=\lim_{x\longrightarrow 0}\frac{\cos x}{1}=1$$
  Como $\lim_{x\longrightarrow 0^-}\dfrac{\sin x}{x}\neq\lim_{x\longrightarrow 0^+}\dfrac{\sin x}{x}$ concluimos con que el límite no existe.
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$$\lim_{x\longrightarrow a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\left[\frac{0}{0}\right]=\lim_{x\longrightarrow a}\frac{\frac{1}{2\sqrt{x}}}{1}=\frac{1}{2\sqrt{a}}$$

f) $\lim_{x\longrightarrow 0}\dfrac{\sin x}{7x^2}=\left[\dfrac{0}{0}\right]=\lim_{x\longrightarrow 0}\dfrac{\cos x}{14x}=\left[\dfrac{1}{0}\right]$
$$\lim_{x\longrightarrow 0^-}\frac{\cos x}{14x}=\left[\frac{1}{0^-}\right]=-\infty$$
$$\lim_{x\longrightarrow 0^+}\frac{\cos x}{14x}=\left[\frac{1}{0^+}\right]=\infty$$
Como $\lim_{x\longrightarrow 0^-}\dfrac{\cos x}{14x}\neq\lim_{x\longrightarrow 0^+}\dfrac{\cos x}{14x}$ concluimos con que el límite no existe.

g)
$$a_n=\frac{2n}{n+1}\Longrightarrow a_{n+1}=\frac{2(n+1)}{n+2}\Longrightarrow n^2(a_n-a_{n+1})=\frac{n^2}{n^2+3n+2}$$
$\lim_{x\longrightarrow\infty}\dfrac{n^2}{n^2+3n+2}=1$