

1° Calcula las razones de 15°

a) a partir de las de 45° y 30°.

b) a partir de las de 30°.

2° Calcula las razones trigonométricas de:

a) 75°

b) 195°

c) 240°

d) 120°

e) 22,5°

f) 345°

g) 7,5°

h) 97,5°

3° Sabiendo que $\operatorname{sen} \alpha = \frac{1}{3}$, $\alpha \in \left(\frac{\pi}{2}, \pi\right)$, calcula de forma exacta $\cos \alpha$, $\operatorname{tg} 2\alpha$ y $\operatorname{sen}\left(\frac{\alpha}{2}\right)$.

4° Sabiendo que $\cos \alpha = \frac{2}{3}$, $270^\circ < \alpha < 360^\circ$, calcula de forma exacta $\operatorname{sen} \alpha$, $\operatorname{sen} \frac{\alpha}{2}$ y $\operatorname{sen} 2\alpha$

5° Identidades trigonométricas:

a) $\operatorname{tg} \alpha + \cot g \alpha = \sec \alpha \cdot \operatorname{cosec} \alpha$

b) $\cot g \alpha \cdot \sec \alpha = \operatorname{cosec} \alpha$

c) $\sec^2 \alpha + \operatorname{cosec}^2 \alpha = \frac{1}{\operatorname{sen}^2 \alpha \cdot \cos^2 \alpha}$

d) $\cot g(\alpha + \beta) = \frac{\cot g \alpha \cdot \cot g \beta - 1}{\cot g \alpha + \cot g \beta}$

e) $\frac{1}{\sec^2 \alpha} = \operatorname{sen}^2 \alpha \cdot \cos^2 \alpha + \cos^4 \alpha$

f) $\operatorname{sen} \beta \cdot \cos(\alpha - \beta) + \cos \beta \cdot \operatorname{sen}(\alpha - \beta) = \operatorname{sen} \alpha$

6° Demuestra que si A, B y C son los ángulos de un triángulo entonces

$$\operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C = \operatorname{tg} A \cdot \operatorname{tg} B \cdot \operatorname{tg} C .$$

7° Demuestra que en un triángulo rectángulo en A se cumple $\operatorname{sen}^2 A = \operatorname{sen}^2 B + \operatorname{sen}^2 C$.

8° En un triángulo ABC demuestra el teorema de Neper o de las tangentes:

$$\frac{a-b}{a+b} = \frac{\operatorname{tg} \frac{A-B}{2}}{\operatorname{tg} \frac{A+B}{2}} .$$

9° Demuestra que $\operatorname{arctg} 1 + \operatorname{arctg} 2 + \operatorname{arctg} 3 = \pi$

10° Simplifica las expresiones:

a) $\frac{\operatorname{sen} 2x}{1 + \cos 2x}$

b) $\frac{\operatorname{sen} 2x}{1 - \cos^2 x} \cdot \frac{\operatorname{sen} 2x}{\cos x}$

c) $\frac{\operatorname{sen} 3x - \operatorname{sen} 5x}{\cos 3x + \cos 5x}$

11° Desarrolla:

a) $\cos(x+y+z)$

b) $\operatorname{sen}(x+y+z)$

① Recuerda

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2} \quad \operatorname{tg} 45^\circ = 1.$$

$$\sin 30^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \operatorname{tg} 30^\circ = \frac{\sqrt{3}}{3}$$

$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$\operatorname{tg}(A-B) = \frac{\operatorname{tg} A - \operatorname{tg} B}{1 + \operatorname{tg} A \cdot \operatorname{tg} B}.$$

$$\begin{aligned} \textcircled{a} \quad \sin 15^\circ &= \sin(45^\circ - 30^\circ) = \sin 45^\circ \cdot \cos 30^\circ - \cos 45^\circ \cdot \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} \cos 15^\circ &= \cos(45^\circ - 30^\circ) = \cos 45^\circ \cdot \cos 30^\circ + \sin 45^\circ \cdot \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

$$\operatorname{tg} 15^\circ = \frac{\operatorname{tg} 45^\circ - \operatorname{tg} 30^\circ}{1 + \operatorname{tg} 45^\circ \cdot \operatorname{tg} 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{\frac{3 - \sqrt{3}}{3}}{\frac{3 + \sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}.$$

Racionalizando:

$$\frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}} = \frac{9 - 6\sqrt{3} + 3}{3^2 - (\sqrt{3})^2} = \frac{12 - 6\sqrt{3}}{6} = 2 - \sqrt{3}$$

También:

$$\begin{aligned} \operatorname{tg} 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} \cdot \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{6 - 2\sqrt{12} + 2}{6 - 2} = \frac{8 - 2\sqrt{12}}{4} \\ &= \frac{8 - 2 \cdot 2\sqrt{3}}{4} = \boxed{2 - \sqrt{3}} \end{aligned}$$

Recuerda

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$\alpha = 30^\circ \rightarrow \frac{\alpha}{2} = 15^\circ$ y seno, coseno y tangente son POSITIVOS.

(b)

$$\sin 15^\circ = \sin \frac{30^\circ}{2} = + \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$
$$\cos 15^\circ = \cos \frac{30^\circ}{2} = + \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$
$$\operatorname{tg} 15^\circ = \operatorname{tg} \frac{30^\circ}{2} = + \sqrt{\frac{1 - \cos 30^\circ}{1 + \cos 30^\circ}} = \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} = \sqrt{7 - 4\sqrt{3}}$$

Racionalizando el radicando:

$$\frac{2 - \sqrt{3}}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{4 - 4\sqrt{3} + 3}{4 - 3} = 7 - 4\sqrt{3}$$

2°

Conocemos las razones trigonométricas de 30° , 45° y 60° . Además por el 1° ejercicio las de 15° .

$$\textcircled{a} \quad \boxed{75^\circ = 30^\circ + 45^\circ} \quad \text{ó} \quad \boxed{75^\circ = 90^\circ - 15^\circ}$$

$$\begin{aligned} \boxed{\operatorname{sen} 75^\circ} &= \operatorname{sen}(30^\circ + 45^\circ) = \operatorname{sen} 30^\circ \cdot \cos 45^\circ + \cos 30^\circ \cdot \operatorname{sen} 45^\circ \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}} \end{aligned}$$

$$\begin{aligned} \boxed{\cos 75^\circ} &= \cos(30^\circ + 45^\circ) = \cos 30^\circ \cdot \cos 45^\circ - \operatorname{sen} 30^\circ \cdot \operatorname{sen} 45^\circ = \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}} \end{aligned}$$

$$\begin{aligned} \boxed{\operatorname{tg} 75^\circ} &= \operatorname{tg}(30^\circ + 45^\circ) = \frac{\operatorname{tg} 30^\circ + \operatorname{tg} 45^\circ}{1 - \operatorname{tg} 30^\circ \cdot \operatorname{tg} 45^\circ} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3} \cdot 1} = \frac{\frac{\sqrt{3} + 3}{3}}{\frac{3 - \sqrt{3}}{3}} \\ &= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} = \frac{9 + 6\sqrt{3} + 3}{9 - 3} = \frac{12 + 6\sqrt{3}}{6} = \boxed{2 + \sqrt{3}} \end{aligned}$$

$$\operatorname{sen} 75^\circ = \operatorname{sen}(90^\circ - 15^\circ) = \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\cos 75^\circ = \cos(90^\circ - 15^\circ) = \operatorname{sen} 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\operatorname{tg} 75^\circ = \operatorname{tg}(90^\circ - 15^\circ) = \cotg 15^\circ = \frac{1}{\operatorname{tg} 15^\circ} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = 2 + \sqrt{3}$$

⑥

$$195^\circ = 180^\circ + 15^\circ$$

ó

$$195^\circ = 270^\circ - 75^\circ$$

Recuerda

$$\operatorname{sen}(180^\circ + \alpha) = -\operatorname{sen} \alpha$$

$$\cos(180^\circ + \alpha) = -\cos \alpha$$

$$\operatorname{tg}(180^\circ + \alpha) = \operatorname{tg} \alpha$$

$$\operatorname{sen} 195^\circ = \operatorname{sen}(180^\circ + 15^\circ) = -\operatorname{sen} 15^\circ = -\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

$$\cos 195^\circ = \cos(180^\circ + 15^\circ) = -\cos 15^\circ = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\operatorname{tg} 195^\circ = \operatorname{tg}(180^\circ + 15^\circ) = \operatorname{tg} 15^\circ = 2-\sqrt{3}$$

Recuerda.

$$\operatorname{sen}(270^\circ - \alpha) = -\cos \alpha$$

$$\cos(270^\circ - \alpha) = -\operatorname{sen} \alpha$$

$$\operatorname{tg}(270^\circ - \alpha) = +\operatorname{ctg} \alpha$$

$$\operatorname{sen} 195^\circ = \operatorname{sen}(270^\circ - 75^\circ) = -\cos 75^\circ = -\cos(90^\circ - 15^\circ) = -\operatorname{sen} 15^\circ$$

$$\cos 195^\circ = \cos(270^\circ - 75^\circ) = -\operatorname{sen} 75^\circ = -\operatorname{sen}(90^\circ - 15^\circ) = -\cos 15^\circ$$

$$\textcircled{c} \quad \boxed{240^\circ = 180^\circ + 60^\circ} \quad \text{ó} \quad \boxed{240^\circ = 270^\circ - 30^\circ}$$

$$\text{sen } 240^\circ = \text{sen } (180^\circ + 60^\circ) = -\text{sen } 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\text{cos } 240^\circ = \text{cos } (180^\circ + 60^\circ) = -\text{cos } 60^\circ = -\frac{1}{2}$$

$$\text{tg } 240^\circ = \frac{\text{sen } (180^\circ + 60^\circ)}{\text{cos } (180^\circ + 60^\circ)} = \frac{-\text{sen } 60^\circ}{-\text{cos } 60^\circ} = \text{tg } 60^\circ = \sqrt{3}$$

$$\text{sen } 240^\circ = \text{sen } (270^\circ - 30^\circ) = -\text{cos } 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\text{cos } 240^\circ = \text{cos } (270^\circ - 30^\circ) = -\text{sen } 30^\circ = -\frac{1}{2}$$

$$\text{tg } 240^\circ = \frac{\text{sen } (270^\circ - 30^\circ)}{\text{cos } (270^\circ - 30^\circ)} = \frac{-\text{cos } 30^\circ}{-\text{sen } 30^\circ} = \text{cotg } 30^\circ = \sqrt{3}$$

$$\textcircled{d} \quad \boxed{120^\circ = 90^\circ + 30^\circ} \quad \text{ó} \quad \boxed{120^\circ = 2 \cdot 60^\circ} \quad \text{ó} \quad \boxed{120^\circ = 180^\circ - 60^\circ}$$

Recomenda

$$\boxed{\begin{aligned} \text{sen } (90^\circ + \alpha) &= \text{cos } \alpha \\ \text{cos } (90^\circ + \alpha) &= -\text{sen } \alpha \\ \text{tg } (90^\circ + \alpha) &= -\text{cotg } \alpha \end{aligned}}$$

$$\bullet \quad \text{sen } 120^\circ = \text{sen } (90^\circ + 30^\circ) = \text{cos } 30^\circ = \frac{\sqrt{3}}{2}$$

$$\text{cos } 120^\circ = \text{cos } (90^\circ + 30^\circ) = -\text{sen } 30^\circ = -\frac{1}{2}$$

$$\text{tg } 120^\circ = \frac{\text{sen } 120^\circ}{\text{cos } 120^\circ} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

$$\begin{aligned} \sin 2\alpha &= 2 \cdot \sin \alpha \cdot \cos \alpha \\ \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ \operatorname{tg} 2\alpha &= \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha} \end{aligned}$$

$$\bullet \sin 120^\circ = \sin 2 \cdot 60^\circ = 2 \cdot \sin 60^\circ \cdot \cos 60^\circ = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 120^\circ = \cos 2 \cdot 60^\circ = \cos^2 60^\circ - \sin^2 60^\circ = \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$\operatorname{tg} 120^\circ = \operatorname{tg} 2 \cdot 60^\circ = \frac{2 \cdot \operatorname{tg} 60^\circ}{1 - \operatorname{tg}^2 60^\circ} = \frac{2 \cdot \sqrt{3}}{1 - (\sqrt{3})^2} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

$$\begin{aligned} \sin(180^\circ - \alpha) &= \sin \alpha \\ \cos(180^\circ - \alpha) &= -\cos \alpha \\ \operatorname{tg}(180^\circ - \alpha) &= -\operatorname{tg} \alpha \end{aligned}$$

$$\bullet \sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$$

$$\operatorname{tg} 120^\circ = \operatorname{tg}(180^\circ - 60^\circ) = -\operatorname{tg} 60^\circ = -\sqrt{3}$$

$$\textcircled{e} \quad 22,5^\circ = \frac{45^\circ}{2}$$

Fórmulas del ángulo mitad

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} \quad \cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}} \quad \operatorname{tg} \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

Observación: como $\alpha \in I \Rightarrow \frac{\alpha}{2} \in I$. y todas las razones trigonométricas son POSITIVAS.

$$\sin 22,5^\circ = \sin \frac{45^\circ}{2} = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}}$$

$$\cos 22,5^\circ = \cos \frac{45^\circ}{2} = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{2}}{4}}$$

$$\operatorname{tg} 22,5^\circ = \operatorname{tg} \frac{45^\circ}{2} = \sqrt{\frac{1 - \cos 45^\circ}{1 + \cos 45^\circ}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$$

$$\textcircled{f} \quad 345^\circ = 360^\circ - 15^\circ \quad \text{Ángulo opuesto}$$

$$\sin(360^\circ - \alpha) = \sin(-\alpha) = -\sin \alpha$$

$$\cos(360^\circ - \alpha) = \cos(-\alpha) = +\cos \alpha$$

$$\operatorname{tg}(360^\circ - \alpha) = \operatorname{tg}(-\alpha) = -\operatorname{tg} \alpha$$

$$\sin 345^\circ = \sin(360^\circ - 15^\circ) = -\sin 15^\circ = -\frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos 345^\circ = \cos(360^\circ - 15^\circ) = \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\operatorname{tg} 345^\circ = \operatorname{tg}(360^\circ - 15^\circ) = -\operatorname{tg} 15^\circ = 2 - \sqrt{3}$$

$$\textcircled{g} \quad 7,5^\circ = \frac{15^\circ}{2}$$

$$\sin 7,5^\circ = \sin \frac{15^\circ}{2} = + \sqrt{\frac{1 - \cos 15^\circ}{2}} = \sqrt{\frac{4 - \sqrt{6} - \sqrt{2}}{8}}$$

$$1 - \cos 15^\circ = 1 - \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{4 - \sqrt{6} - \sqrt{2}}{4} \quad \uparrow$$

$$\cos 7,5^\circ = \cos \frac{15^\circ}{2} = + \sqrt{\frac{1 + \cos 15^\circ}{2}} = \sqrt{\frac{4 + \sqrt{6} + \sqrt{2}}{8}}$$

$$1 + \cos 15^\circ = 1 + \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{4 + \sqrt{6} + \sqrt{2}}{4}$$

$$\tan 7,5^\circ = \frac{\sin 7,5^\circ}{\cos 7,5^\circ} = \sqrt{\frac{4 - \sqrt{6} - \sqrt{2}}{4 + \sqrt{6} + \sqrt{2}}}$$

$$\textcircled{h} \quad 97,5^\circ = 90^\circ + 7,5^\circ$$

Obtén las fórmulas del apartado \textcircled{g} .

$$\sin 97,5^\circ = \sin (90^\circ + 7,5^\circ) = \cos 7,5^\circ = \sqrt{\frac{4 + \sqrt{6} + \sqrt{2}}{8}}$$

$$\cos 97,5^\circ = \cos (90^\circ + 7,5^\circ) = -\sin 7,5^\circ = -\sqrt{\frac{4 - \sqrt{6} - \sqrt{2}}{8}}$$

$$\tan 97,5^\circ = \frac{\sin 97,5^\circ}{\cos 97,5^\circ} = -\sqrt{\frac{4 + \sqrt{6} + \sqrt{2}}{4 - \sqrt{6} - \sqrt{2}}}$$

3°

¿ $\cos \alpha$?

Relación fundamental de la trigonometría

$$\sin^2 \alpha + \cos^2 \alpha = 1 \Leftrightarrow \left(\frac{1}{3}\right)^2 + \cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\Rightarrow \cos \alpha = \pm \sqrt{\frac{8}{9}} = \boxed{-\frac{\sqrt{8}}{3}} \quad (\alpha \in \text{II}, \cos \alpha < 0)$$

¿ $\sin \frac{\alpha}{2}$?

Fórmula $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$

• signo. $\alpha \in \text{II} \Leftrightarrow \frac{\pi}{2} < \alpha < \pi \Leftrightarrow \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$

$$\Rightarrow \frac{\alpha}{2} \in \text{I} \rightarrow \sin \frac{\alpha}{2} > 0.$$

• valor: $\sin \frac{\alpha}{2} = + \sqrt{\frac{1 - (-\frac{\sqrt{8}}{3})}{2}} = \boxed{+ \sqrt{\frac{3 + \sqrt{8}}{6}}}$

¿ $\tan 2\alpha$?

Fórmula $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{1/3}{-\sqrt{8}/3} = -\frac{1}{\sqrt{8}} = \frac{-\sqrt{8}}{8} = \frac{-2 \cdot \sqrt{2}}{8} = -\frac{\sqrt{2}}{4}$$

$$\Rightarrow \tan 2\alpha = \frac{2 \cdot \frac{-\sqrt{2}}{4}}{1 - \left(\frac{-\sqrt{2}}{4}\right)^2} = \frac{\frac{-\sqrt{2}}{2}}{1 - \frac{2}{16}} = \frac{\frac{-\sqrt{2}}{2}}{\frac{14}{16}} = -\frac{16\sqrt{2}}{14 \cdot 2} = \boxed{-\frac{2\sqrt{2}}{7}}$$

4º ¿sen α?

Relación fundamental de la trigonometría.

$$\cos^2 \alpha + \sin^2 \alpha = 1 \Leftrightarrow \left(\frac{2}{3}\right)^2 + \sin^2 \alpha = 1 \Rightarrow \sin^2 \alpha = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \sin \alpha = \pm \sqrt{\frac{5}{9}} =$$

¿signo? $\alpha \in \text{IV} \Rightarrow \sin \alpha < 0 \Rightarrow \boxed{\sin \alpha = -\frac{\sqrt{5}}{3}}$

¿sen $\frac{\alpha}{2}$?

Fórmula $\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$

¿signo? $\alpha \in \text{IV} \Leftrightarrow 270^\circ < \alpha < 360^\circ \rightarrow \frac{270}{2} < \frac{\alpha}{2} < \frac{360}{2}$

$\Leftrightarrow 135^\circ < \frac{\alpha}{2} < 180^\circ \Leftrightarrow \frac{\alpha}{2} \in \text{II} \Rightarrow \sin \frac{\alpha}{2} > 0$

$$\Rightarrow \boxed{\sin \frac{\alpha}{2} = + \sqrt{\frac{1 - 2/3}{2}} = \sqrt{\frac{1/3}{2}} = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}}$$

¿sen 2α?

Fórmula. $\sin 2\alpha = 2 \cdot \sin \alpha \cdot \cos \alpha$

$$\boxed{\sin 2\alpha = 2 \cdot \frac{-\sqrt{5}}{3} \cdot \frac{2}{3} = \frac{-4\sqrt{5}}{9}}$$

5°

a) $\boxed{\operatorname{tg} \alpha + \operatorname{ctg} \alpha = \sec \alpha \cdot \operatorname{cosec} \alpha}$

$$\Rightarrow \frac{\operatorname{sen} \alpha}{\operatorname{cos} \alpha} + \frac{\operatorname{cos} \alpha}{\operatorname{sen} \alpha} = \frac{\operatorname{sen}^2 \alpha + \operatorname{cos}^2 \alpha}{\operatorname{cos} \alpha \cdot \operatorname{sen} \alpha} = \frac{1}{\operatorname{cos} \alpha \cdot \operatorname{sen} \alpha} = \frac{1}{\operatorname{cos} \alpha} \cdot \frac{1}{\operatorname{sen} \alpha}$$

$$\begin{aligned} \Leftarrow \frac{1}{\operatorname{cos} \alpha} \cdot \frac{1}{\operatorname{sen} \alpha} &= \frac{\operatorname{sen}^2 \alpha + \operatorname{cos}^2 \alpha}{\operatorname{cos} \alpha \cdot \operatorname{sen} \alpha} = \frac{\operatorname{sen}^2 \alpha}{\operatorname{cos} \alpha \cdot \operatorname{sen} \alpha} + \frac{\operatorname{cos}^2 \alpha}{\operatorname{cos} \alpha \cdot \operatorname{sen} \alpha} \\ &= \frac{\operatorname{sen} \alpha}{\operatorname{cos} \alpha} + \frac{\operatorname{cos} \alpha}{\operatorname{sen} \alpha} \end{aligned}$$

b) $\boxed{\operatorname{ctg} \alpha \cdot \sec \alpha = \operatorname{cosec} \alpha}$

$$\Rightarrow \frac{\operatorname{cos} \alpha}{\operatorname{sen} \alpha} \cdot \frac{1}{\operatorname{cos} \alpha} = \frac{1}{\operatorname{sen} \alpha} = \operatorname{cosec} \alpha$$

$$\Leftarrow \operatorname{cosec} \alpha = \frac{1}{\operatorname{sen} \alpha} = \frac{1}{\operatorname{sen} \alpha} \cdot \frac{\operatorname{cos} \alpha}{\operatorname{cos} \alpha} = \frac{\operatorname{cos} \alpha}{\operatorname{sen} \alpha} \cdot \frac{1}{\operatorname{cos} \alpha}$$

c) $\boxed{\sec^2 \alpha + \operatorname{cosec}^2 \alpha = \frac{1}{\operatorname{sen}^2 \alpha \cdot \operatorname{cos}^2 \alpha}}$

$$\Rightarrow \frac{1}{\operatorname{cos}^2 \alpha} + \frac{1}{\operatorname{sen}^2 \alpha} = \frac{\operatorname{sen}^2 \alpha + \operatorname{cos}^2 \alpha}{\operatorname{cos}^2 \alpha \cdot \operatorname{sen}^2 \alpha} = \frac{1}{\operatorname{cos}^2 \alpha \cdot \operatorname{sen}^2 \alpha}$$

$$\Leftarrow \frac{1}{\operatorname{sen}^2 \alpha \cdot \operatorname{cos}^2 \alpha} = \frac{\operatorname{sen}^2 \alpha + \operatorname{cos}^2 \alpha}{\operatorname{sen}^2 \alpha \cdot \operatorname{cos}^2 \alpha} = \frac{\operatorname{sen}^2 \alpha}{\operatorname{sen}^2 \alpha \cdot \operatorname{cos}^2 \alpha} + \frac{\operatorname{cos}^2 \alpha}{\operatorname{sen}^2 \alpha \cdot \operatorname{cos}^2 \alpha}$$

$$= \frac{1}{\operatorname{cos}^2 \alpha} + \frac{1}{\operatorname{sen}^2 \alpha} = \sec^2 \alpha + \operatorname{cosec}^2 \alpha$$

$$d) \quad \boxed{\cot(\alpha + \beta) = \frac{\cot \alpha \cdot \cot \beta - 1}{\cot \alpha + \cot \beta}}$$

$$\Rightarrow \cot(\alpha + \beta) = \frac{1}{\tan(\alpha + \beta)} = \frac{1 - \tan \alpha \cdot \tan \beta}{\tan \alpha + \tan \beta} = \frac{1 - \frac{1}{\cot \alpha} \cdot \frac{1}{\cot \beta}}{\frac{1}{\cot \alpha} + \frac{1}{\cot \beta}}$$

$$= \frac{\frac{\cot \alpha \cdot \cot \beta - 1}{\cot \alpha \cdot \cot \beta}}{\frac{\cot \beta + \cot \alpha}{\cot \alpha \cdot \cot \beta}} = \frac{\cot \alpha \cdot \cot \beta - 1}{\cot \alpha + \cot \beta} \cdot \cdot$$

$$\Leftarrow \frac{\frac{1}{\tan \alpha} \cdot \frac{1}{\tan \beta} - 1}{\frac{1}{\tan \alpha} + \frac{1}{\tan \beta}} = \frac{\frac{1 - \tan \alpha \cdot \tan \beta}{\tan \alpha \cdot \tan \beta}}{\frac{\tan \beta + \tan \alpha}{\tan \alpha \cdot \tan \beta}} = \frac{1 - \tan \alpha \cdot \tan \beta}{\tan \alpha + \tan \beta}$$

$$= \frac{1}{\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}} = \frac{1}{\tan(\alpha + \beta)} = \cot(\alpha + \beta)$$

$$c) \boxed{\frac{1}{\sec^2 \alpha} = \sin^2 \alpha \cdot \cos^2 \alpha + \cos^4 \alpha}$$

$$\Rightarrow \frac{1}{\sec^2 \alpha} = \cos^2 \alpha = \cos^2 \alpha \cdot 1 = \cos^2 \alpha \cdot (\sin^2 \alpha + \cos^2 \alpha)$$

$$= \cos^2 \alpha \cdot \sin^2 \alpha + \cos^4 \alpha \quad \bullet$$

$$\Leftarrow \sin^2 \alpha \cdot \cos^2 \alpha + \cos^4 \alpha = \cos^2 \alpha \cdot (\sin^2 \alpha + \cos^2 \alpha) = \cos^2 \alpha$$

$$= \frac{1}{\sec^2 \alpha} \quad \bullet$$

$$f) \boxed{\sin \beta \cdot \cos(\alpha - \beta) + \cos \beta \cdot \sin(\alpha - \beta) = \sin \alpha}$$

$$\Rightarrow \sin \beta \cdot (\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta) + \cos \beta \cdot (\sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta) =$$

$$\underbrace{\sin \beta \cdot \cos \alpha \cdot \cos \beta}_{\sin^2 \beta \cdot \cos \alpha} + \sin \beta \cdot \sin \alpha + \underbrace{\cos^2 \beta \cdot \sin \alpha}_{\cos^2 \beta \cdot \sin \alpha} - \underbrace{\cos \beta \cdot \cos \alpha \cdot \sin \beta}_{\sin \beta \cdot \cos \alpha \cdot \cos \beta} =$$

$$\underbrace{\sin^2 \beta \cdot \cos \alpha}_{\sin^2 \beta \cdot \cos \alpha} + \underbrace{\cos^2 \beta \cdot \sin \alpha}_{\cos^2 \beta \cdot \sin \alpha} = (\sin^2 \beta + \cos^2 \beta) \cdot \sin \alpha = \sin \alpha \quad \bullet$$

$$\Leftarrow \sin \alpha = \sin \alpha \cdot (\sin^2 \beta + \cos^2 \beta) = \sin \alpha \cdot \sin^2 \beta + \sin \alpha \cdot \cos^2 \beta$$

$$= \sin \alpha \cdot \sin^2 \beta + \sin \alpha \cdot \cos^2 \beta + \sin \beta \cdot \cos \alpha \cdot \cos \beta - \sin \beta \cdot \cos \alpha \cdot \cos \beta =$$

$$= \sin \beta (\sin \alpha \cdot \sin \beta + \cos \alpha \cdot \cos \beta) + \cos \beta (\sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta)$$

$$= \sin \beta \cdot \cos(\alpha + \beta) + \cos \beta \cdot \sin(\alpha - \beta) \quad \bullet$$

8º Fórmulas de Neper.

El teorema del seno establece

$$\frac{a}{\operatorname{sen} A} = \frac{b}{\operatorname{sen} B} \rightarrow$$

$$\boxed{\frac{a}{b} = \frac{\operatorname{sen} A}{\operatorname{sen} B}}$$

$$\Rightarrow \frac{a-b}{a+b} = \frac{a}{a+b} - \frac{b}{a+b} = \frac{1}{1 + \frac{b}{a}} - \frac{1}{1 + \frac{a}{b}}$$

$$= \frac{1}{1 + \frac{\operatorname{sen} B}{\operatorname{sen} A}} - \frac{1}{1 + \frac{\operatorname{sen} A}{\operatorname{sen} B}} = \frac{\operatorname{sen} A}{\operatorname{sen} A + \operatorname{sen} B} - \frac{\operatorname{sen} B}{\operatorname{sen} B + \operatorname{sen} A}$$

$$= \frac{\operatorname{sen} A - \operatorname{sen} B}{\operatorname{sen} A + \operatorname{sen} B} = \frac{2 \cdot \operatorname{sen} \frac{A+B}{2} \cdot \operatorname{sen} \frac{A-B}{2}}{2 \cdot \operatorname{sen} \frac{A+B}{2} \cdot \operatorname{sen} \frac{A-B}{2}} = \frac{\operatorname{tg} \frac{A-B}{2}}{\operatorname{tg} \frac{A+B}{2}}$$

$$\Leftarrow \frac{\operatorname{tg} \frac{A-B}{2}}{\operatorname{tg} \frac{A+B}{2}} = \frac{\operatorname{sen} \frac{A-B}{2}}{\operatorname{sen} \frac{A+B}{2}} \cdot \frac{\operatorname{tg} \frac{A+B}{2}}{\operatorname{sen} \frac{A+B}{2}} = \frac{\frac{1}{2} \cdot (\operatorname{sen} A - \operatorname{sen} B)}{\frac{1}{2} \cdot (\operatorname{sen} A + \operatorname{sen} B)} =$$

$$= \frac{\operatorname{sen} A - \operatorname{sen} B}{\operatorname{sen} A + \operatorname{sen} B} = \frac{\operatorname{sen} A \cdot \left(1 - \frac{\operatorname{sen} B}{\operatorname{sen} A}\right)}{\operatorname{sen} A \cdot \left(1 + \frac{\operatorname{sen} B}{\operatorname{sen} A}\right)} = \frac{1 - \frac{b}{a}}{1 + \frac{b}{a}} = \frac{a-b}{a+b}$$

RECUERDA.

$$\operatorname{sen} A + \operatorname{sen} B = 2 \cdot \operatorname{sen} \frac{A+B}{2} \cdot \operatorname{sen} \frac{A-B}{2}$$

$$\operatorname{sen} A - \operatorname{sen} B = 2 \cdot \operatorname{sen} \frac{A+B}{2} \cdot \operatorname{sen} \frac{A-B}{2}$$

$$\textcircled{P} \quad \boxed{\arctan 1 + \arctan 2 + \arctan 3 = \pi.}$$

$$\alpha = \arctan 1 \Leftrightarrow \tan \alpha = 1.$$

$$\beta = \arctan 2 \Leftrightarrow \tan \beta = 2$$

$$\gamma = \arctan 3 \Leftrightarrow \tan \gamma = 3$$

(1) • supomos que $\alpha, \beta, \gamma \in (0, \pi/2)$

$$(2) \cdot \boxed{\arctan x + \arctan y = \arctan \left(\frac{x+y}{1-xy} \right)}$$

$$\tan A = x \Leftrightarrow A = \arctan x$$

$$\tan B = y \Leftrightarrow B = \arctan y$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \Leftrightarrow \arctan [\tan(A+B)] = \arctan \left[\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \right]$$

$$\Leftrightarrow A+B = \arctan \left[\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \right]$$

$$\Leftrightarrow \arctan x + \arctan y = \arctan \left(\frac{x+y}{1-x \cdot y} \right)$$

Aplicando esta fórmula a nuestro problema

$$\arctan 1 + \arctan 2 = \arctan \left(\frac{1+2}{1-1 \cdot 2} \right) = \arctan(-3)$$

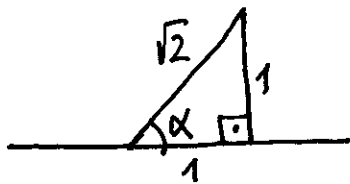
$$\underbrace{\arctan 1 + \arctan 2}_{\uparrow} + \arctan 3 = \arctan(-3) + \arctan(3) = \arctan$$

$$= \arctan \left(\frac{-3+3}{1-(-3) \cdot 3} \right) = \arctan 0 = k\pi, \quad k \in \mathbb{Z}.$$

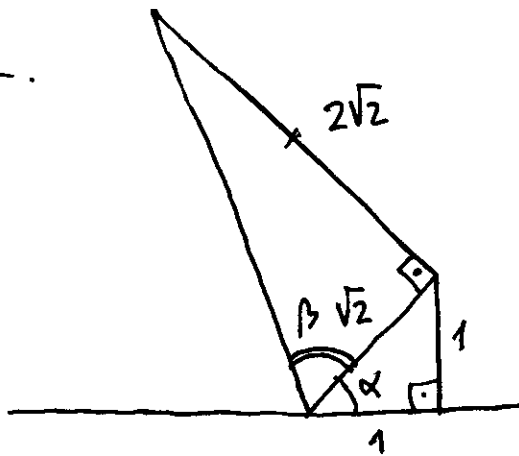
Como $\alpha + \beta + \gamma > 0$ y $\alpha + \beta + \gamma \leq 3 \frac{\pi}{2} \stackrel{(1)}{\Rightarrow} k = 1$. Queda demostrado.

$$\Rightarrow \arctan 1 + \arctan 2 + \arctan 3 = \pi.$$

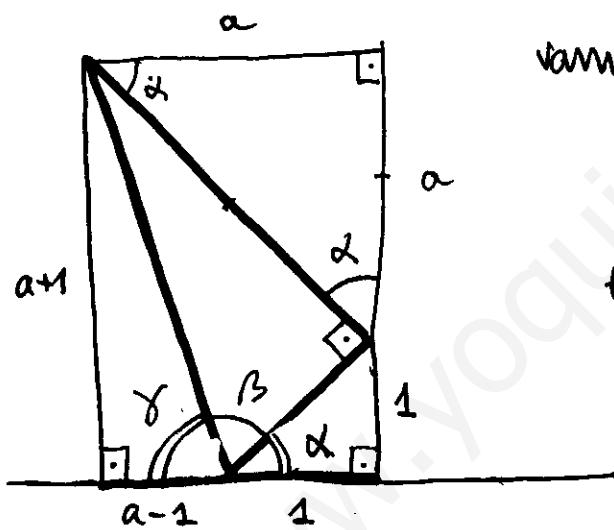
Prueba de forma grafica.



$$\begin{aligned} \operatorname{tg} \alpha &= 1. \\ \alpha &= \arctg 1 \end{aligned}$$



$$\begin{aligned} \operatorname{tg} \beta &= \frac{2\sqrt{2}}{\sqrt{2}} = 2 \\ \beta &= \arctg 2. \end{aligned}$$



vamos a probar que γ cumple que $\operatorname{tg} \gamma = 3$

$$\Rightarrow \boxed{\alpha + \beta + \gamma = \pi}$$

$$\operatorname{tg} \gamma = \frac{a+1}{a-1}.$$

¿a?

$$a^2 + a^2 = (2\sqrt{2})^2.$$

$$2a^2 = 8 \rightarrow a^2 = 4 \rightarrow a = 2$$

$$\Rightarrow \operatorname{tg} \gamma = \frac{2+1}{2-1} = 3$$

10°

$$\begin{aligned} \text{a) } \frac{\sin 2x}{1 + \cos 2x} &= \frac{2 \cdot \sin x \cdot \cos x}{\underbrace{1 + \cos^2 x - \sin^2 x}} = \frac{2 \cdot \sin x \cdot \cos x}{\underbrace{\cos^2 x + \sin^2 x + \cos^2 x - \sin^2 x}} \\ &= \frac{2 \cdot \sin x \cdot \cos x}{2 \cdot \cos^2 x} = \frac{\sin x}{\cos x} = \operatorname{tg} x \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{\sin 2x}{1 - \cos^2 x} \cdot \frac{\sin 2x}{\cos x} &= \frac{2 \cdot \sin x \cdot \cos x}{\sin^2 x} \cdot \frac{2 \cdot \sin x \cdot \cos x}{\cos x} = \\ &= \frac{4 \cdot \sin^2 x \cdot \cos^2 x}{\sin^2 x \cdot \cos x} = 4 \cdot \cos x \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{\sin 3x - \sin 5x}{\cos 3x + \cos 5x} &= \frac{2 \cdot \cos \frac{3x+5x}{2} \cdot \sin \frac{3x-5x}{2}}{2 \cdot \cos \frac{3x+5x}{2} \cdot \cos \frac{3x-5x}{2}} \\ &= \frac{\sin(-x)}{\cos(-x)} = \operatorname{tg}(-x) = -\operatorname{tg} x \end{aligned}$$

11º Fórmulas

$$\left. \begin{aligned} \cos(A+B) &= \cos A \cdot \cos B - \sin A \cdot \sin B \\ \cos(A-B) &= \cos A \cdot \cos B + \sin A \cdot \sin B \end{aligned} \right\}$$

$$\left. \begin{aligned} \sin(A+B) &= \sin A \cdot \cos B + \cos A \cdot \sin B \\ \sin(A-B) &= \sin A \cdot \cos B - \cos A \cdot \sin B \end{aligned} \right\}$$

$$\begin{aligned} \bullet \cos [x+y+z] &= \cos [(x+y)+z] = \\ &= \cos(x+y) \cdot \cos z - \sin(x+y) \cdot \sin z = \\ &= (\cos x \cdot \cos y + \sin x \sin y) \cos z - (\sin x \cdot \cos y + \cos x \cdot \sin y) \cdot \sin z \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \cos(x+y+z) &= \cos x \cdot \cos y \cdot \cos z + \sin x \cdot \sin y \cdot \cos z \\ &\quad - \sin x \cdot \cos y \cdot \sin z - \cos x \cdot \sin y \cdot \sin z \end{aligned}}$$

$$\begin{aligned} \bullet \sin [x+y+z] &= \sin [(x+y)+z] = \\ &= \sin(x+y) \cdot \cos z + \cos(x+y) \cdot \sin z = \\ &= (\sin x \cdot \cos y + \cos x \cdot \sin y) \cdot \cos z + (\cos x \cdot \cos y - \sin x \sin y) \sin z \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \sin(x+y+z) &= \sin x \cdot \cos y \cdot \cos z + \cos x \sin y \cos z + \\ &\quad \cos x \cdot \cos y \cdot \sin z - \sin x \cdot \sin y \cdot \sin z. \end{aligned}}$$

⑥ Recuerda

$$\bullet \operatorname{tg}(A+B) = \frac{\operatorname{tg}A + \operatorname{tg}B}{1 - \operatorname{tg}A \cdot \operatorname{tg}B}$$

$$\Rightarrow \operatorname{tg}A + \operatorname{tg}B = \operatorname{tg}(A+B) \cdot [1 - \operatorname{tg}A \cdot \operatorname{tg}B]$$

• en un triángulo $A+B+C=180$

$$\bullet \operatorname{tg}(180-A) = -\operatorname{tg}A$$

Demostración:

$$\operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C = \operatorname{tg}(A+B) \cdot [1 - \operatorname{tg}A \cdot \operatorname{tg}B] + \operatorname{tg}C$$

$$= \operatorname{tg}(180-C) \cdot [1 - \operatorname{tg}A \cdot \operatorname{tg}B] + \operatorname{tg}C$$

$$= -\operatorname{tg}C \cdot [1 - \operatorname{tg}A \cdot \operatorname{tg}B] + \operatorname{tg}C$$

$$= -\cancel{\operatorname{tg}C} + \operatorname{tg}C \cdot \operatorname{tg}A \cdot \operatorname{tg}B + \cancel{\operatorname{tg}C}$$

$$\Rightarrow \boxed{\operatorname{tg}A + \operatorname{tg}B + \operatorname{tg}C = \operatorname{tg}A \cdot \operatorname{tg}B \cdot \operatorname{tg}C}$$

⑦ En un triángulo rectángulo en A.

$$A = 90^\circ \rightarrow \operatorname{sen}A = 1.$$

$$B+C=90^\circ \rightarrow \operatorname{sen}B = \operatorname{cos}C \quad \text{y} \quad \operatorname{cos}B = \operatorname{sen}C$$

relación fundamental de la trigonometría para B

$$1 = \operatorname{sen}^2B + \operatorname{cos}^2B$$

$$1 = \operatorname{sen}^2B + \operatorname{sen}^2C$$

$$\boxed{\operatorname{sen}^2A = \operatorname{sen}^2B + \operatorname{sen}^2C}$$