ACTIVIDADES UNIDAD 2: Números complejos

El camino más corto entre dos verdades del
Análisis Real pasa por el Análisis Complejo.
Jacques Hadamard (1865-1963)

1. Dados los siguientes números complejos:

$$z_1 = 4-5i \quad z_2 = 2+3i \quad z_3 = -3+5i \quad z_4 = 6+2i$$
$$z_5 = (7,8) \quad z_6 = (-4,-9) \quad z_7 = (-12,2) \quad z_8 = (4,5)$$

efectúa las siguientes operaciones algebraicas:

1) $z_1 + z_2$ 2) $z_4 - z_3$ 3) $\dfrac{z_5 \cdot z_3}{z_6}$ 4) $\left(z_8 + z_7\right) \cdot z_4$

2. Escribe en forma polar el resultado del cociente: $\dfrac{i^5 - i^{-8}}{i\sqrt{2}}$

3. Dados los números complejos $z_1 = 5_{\pi/4}$, $z_2 = 2_{15^\circ}$ y $z_3 = 4i$, calcula

a) $z_3 \cdot z_2$ b) $\dfrac{z_1}{(z_2)^2}$ c) $\dfrac{z_1 \cdot z_2^3}{z_3}$ d) $\dfrac{(z_1)^3}{z_2 \cdot (z_3)^2}$

4. Sea $z = \sqrt{3} - i$. Calcular: a) $\overline{z}$ b) $\dfrac{1}{z}$ c) z^4 d) $\sqrt[4]{z}$

5. Expresa en forma polar:

a) $4-3i$ b) $5+12i$ c) $-3+3i$ d) $-2-4i$

6. Expresa en forma trigonométrica los complejos:

a) $-3+3\sqrt{3}i$ b) $1-i$ c) $6-5i$ d) $-9-8i$

7. Expresa en forma binómica los siguientes complejos:

a) 7_{120° b) $2_{\pi/6}$ c) $3_{3\pi/4}$ d) 5_{135°

8. Realiza las operaciones en forma polar y después pasa a forma binómica:

a) $3_{45^\circ} \cdot 2_{15^\circ}$ b) $6_{-21^\circ} : 2_{24^\circ}$ c) $(2_{25^\circ})^3 \cdot 3_{15^\circ}$ d) $\left(\sqrt{2} - i\right)^6$

9. Halla las siguientes raíces:

a) $\sqrt[3]{1+i}$ b) $\sqrt[3]{-i}$ c) $\sqrt[6]{-64}$ d) $\sqrt[3]{-27}$

10. Calcula:

a) $\sqrt[5]{\dfrac{32}{-i}}$ b) $\left(\dfrac{i^5 - i^{-8}}{\sqrt{2}i}\right)^5$ c) $\left(\dfrac{1+i}{2-i}\right)^5$ d) $\sqrt[4]{-8+8\sqrt{3}i}$

1) 1) $z_1 + z_2 = (4-5i) + (2+3i) = \boxed{6-2i}$

2) $z_4 - z_3 = (6+2i) - (-3+5i) = \boxed{9-3i}$

3) $$\frac{z_5 \cdot z_3}{z_6} = \frac{(7,8)\cdot(-3+5i)}{(-4,-9)} = \frac{(7+8i)\cdot(-3+5i)}{-4-9i} = \frac{-21+35i-24i-40}{-4-9i} =$$

$$= \frac{-61+11i}{-4-9i} = \frac{(-61+11i)(-4+9i)}{(-4-9i)(-4+9i)} = \frac{244-549i-44i-99}{16-36i+36i+81} = \frac{145-593i}{97} =$$

$$= \boxed{\frac{145}{97} - \frac{593}{97}i}$$

4) $$(z_8+z_7)\cdot z_4 = [(4,5)+(-12,2)]\cdot(6+2i) = (-8,7)\cdot(6+2i) = (-8+7i)\cdot(6+2i) =$$

$$= -48-16i+42i-14 = \boxed{-62+26i}$$

2) $$z = \frac{i^5 - i^{-8}}{i\sqrt{2}} = \frac{i - \frac{1}{i^8}}{i\sqrt{2}} = \frac{i - \frac{1}{i^0}}{i\sqrt{2}} = \frac{i - \frac{1}{1}}{i\sqrt{2}} = \frac{i-1}{i\sqrt{2}} = \frac{(i-1)(-i\sqrt{2})}{(i\sqrt{2})(-i\sqrt{2})} =$$

$$= \frac{\sqrt{2}+i\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

Módulo: $|z| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = \sqrt{\frac{4}{4}} = 1$

Argumento: $\alpha = \operatorname{arctg} \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \operatorname{arctg} 1 = 45°$

Por tanto, $\boxed{z = 1_{45°}}$

3) Como las operaciones en forma polar son muy sencillas, escribimos todos los nºs complejos en dicha forma:

$z_1 = 5_{\frac{\pi}{4}} = 5_{45°}$, $z_2 = 2_{15°}$, $z_3 = 4i = 4_{90°}$

a) $z_3 \cdot z_2 = 4_{90°} \cdot 2_{15°} = (4\cdot 2)_{90°+15°} = \boxed{8_{105°}}$

b) $$\frac{z_1}{z_2^2} = \frac{5_{45°}}{(2_{15°})^2} = \frac{5_{45°}}{4_{15°\cdot 2}} = \frac{5_{45°}}{4_{30°}} = \left(\frac{5}{4}\right)_{45°-30°} = \boxed{\left(\frac{5}{4}\right)_{15°}}$$

c) $$\frac{z_1 \cdot z_2^3}{z_3} = \frac{5_{45°}\cdot(2_{15°})^3}{4_{90°}} = \frac{5_{45°}\cdot 8_{45°}}{4_{90°}} = \frac{40_{90°}}{4_{90°}} = \left(\frac{40}{4}\right)_{90°-90°} = \boxed{10}$$

d) $$\frac{z_1^3}{z_2 \cdot z_3^2} = \frac{(5_{45°})^3}{2_{15°}\cdot(4_{90°})^2} = \frac{125_{135°}}{2_{15°}\cdot 16_{180°}} = \frac{125_{135°}}{32_{195°}} = \left(\frac{125}{32}\right)_{-60°} = \boxed{\left(\frac{125}{32}\right)_{300°}}$$

④ En los apartados a) y b) trabajaremos en forma binómica y en los apartados c) y d) en forma polar:

a) $\bar{z} = \boxed{\sqrt{3}+i}$

b) $\frac{1}{z} = \frac{1}{\sqrt{3}-i} = \frac{\sqrt{3}+i}{(\sqrt{3}-i)(\sqrt{3}+i)} = \frac{\sqrt{3}+i}{3+\sqrt{3}i-\sqrt{3}i+1} = \frac{\sqrt{3}+i}{4} = \boxed{\frac{\sqrt{3}}{4}+\frac{1}{4}i}$

c) $z = \sqrt{3}-i$

Módulo : $|z| = \sqrt{(\sqrt{3})^2+(-1)^2} = 2$

Argumento : $\alpha = 360^\circ + \text{arctg}\,\frac{-1}{\sqrt{3}} = 360^\circ - 30^\circ = 330^\circ$

$\Big\}\ z = 2_{330^\circ}$

$z^4 = (2_{330^\circ})^4 = (2^4)_{4\cdot 330^\circ} = 16_{1320} = \boxed{16_{240^\circ}}$

d)

$$\sqrt[4]{z} = \sqrt[4]{2_{330^\circ}} = \boxed{\begin{cases} \sqrt[4]{2}_{\beta_1} = \sqrt[4]{2}_{82{,}5^\circ} \\ \sqrt[4]{2}_{\beta_2} = \sqrt[4]{2}_{172{,}5^\circ} \\ \sqrt[4]{2}_{\beta_3} = \sqrt[4]{2}_{262{,}5^\circ} \\ \sqrt[4]{2}_{\beta_4} = \sqrt[4]{2}_{352{,}5^\circ} \end{cases}}$$

$\beta_1 = \frac{330^\circ+0\cdot 360^\circ}{4} = 82{,}5^\circ$, $\beta_2 = \frac{330^\circ+1\cdot 360^\circ}{4} = 262{,}5^\circ$

$\beta_3 = \frac{330^\circ+2\cdot 360^\circ}{4} = 172{,}5^\circ$, $\beta_4 = \frac{330^\circ+3\cdot 360^\circ}{4} = 352{,}5^\circ$

⑤ a) $z_1 = 4-3i$

Módulo : $|z_1| = \sqrt{4^2+(-3)^2} = 5$

Argumento : $\alpha_1 = 360^\circ + \text{arctg}\,\frac{-3}{4} = 360^\circ - 37^\circ = 323^\circ$

$\Big\}\ \boxed{z_1 = 5_{323^\circ}}$

b) $z_2 = 5+12i$

Módulo : $|z_2| = \sqrt{5^2+12^2} = 13$

Argumento : $\alpha_2 = \text{arctg}\,\frac{12}{5} \simeq 67^\circ$

$\Big\}\ \boxed{z_2 = 13_{67^\circ}}$

c) $z_3 = -3+3i$

Módulo : $|z_3| = \sqrt{(-3)^2+3^2} = \sqrt{18}$

Argumento : $\alpha_3 = 180^\circ + \text{arctg}\left(\frac{3}{-3}\right) = 180^\circ - 45^\circ = 135^\circ$

$\Big\}\ \boxed{z_3 = \sqrt{18}_{135^\circ}}$

d) $z_4 = -2-4i$

Módulo : $|z_4| = \sqrt{(-2)^2+(-4)^2} = \sqrt{20}$

Argumento : $\alpha_4 = 180° + \text{arctg}\,\frac{-4}{-2} = 180° + 63° \simeq 243°$

$\left.\right\} \boxed{z_4 = \sqrt{20}_{\,243°}}$

6 a) $z_1 = -3+3\sqrt{3}\,i$

Módulo : $|z_1| = \sqrt{(-3)^2+(3\sqrt{3})^2} = 6$

Argumento : $\alpha_1 = 180° + \text{arctg}\,\frac{3\sqrt{3}}{-3} = 180° - 60° = 120°$

$z_1 = 6_{120°} = \boxed{6\,(\cos 120° + i \,\text{sen}\, 120°)}$

b) $z_2 = 1-i$

Módulo : $|z_2| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

Argumento : $\alpha_2 = 360° + \text{arctg}\,\frac{-1}{1} = 360° - 45° = 315°$

$z_2 = \sqrt{2}_{\,315°} = \boxed{\sqrt{2}\,(\cos 315° + i\,\text{sen}\, 315°)}$

c) $z_3 = 6-5i$

Módulo : $|z_3| = \sqrt{6^2+(-5)^2} = \sqrt{61}$

Argumento : $\alpha_3 = 360° + \text{arctg}\,\frac{-5}{6} \simeq 360° - 40° = 320°$

$z_3 = \sqrt{61}_{\,320°} = \boxed{\sqrt{61}\,(\cos 320° + i\,\text{sen}\, 320°)}$

d) $z_4 = -9-8i$

Módulo : $|z_4| = \sqrt{(-9)^2+(-8)^2} = \sqrt{145}$

Argumento : $\alpha_4 = 180° + \text{arctg}\left(\frac{-8}{-9}\right) \simeq 180° + 42° = 222°$

$z_4 = \sqrt{145}_{\,222°} = \boxed{\sqrt{145}\,(\cos 222° + i\,\text{sen}\, 222°)}$

7 a) $z_1 = 7_{120°} = 7\,(\cos 120° + i\,\text{sen}\, 120°) = 7\cos 120° + i\,(7\,\text{sen}\, 120°) =$

$= \boxed{-3,5 + 6,1\,i}$

b) $z_2 = 2_{\pi/6} = 2_{30°} = 2\,(\cos 30° + i\,\text{sen}\, 30°) = 2\cos 30° + i\,(2\,\text{sen}\, 30°) =$

$= \boxed{1,7 + i}$

c) $z_3 = 3_{\frac{3\pi}{4}} = 3_{135^\circ} = 3(\cos 135^\circ + i \,\mathrm{sen}\, 135^\circ) = 3\cos 135^\circ + i(3\,\mathrm{sen}\,135^\circ) =$

$= \boxed{-2{,}1 + 2{,}1i}$

d) $z_4 = 5_{135^\circ} = 5(\cos 135^\circ + i\,\mathrm{sen}\,135^\circ) = 5\cos 135^\circ + i(5\,\mathrm{sen}\,135^\circ) =$

$= \boxed{-3{,}5 + 3{,}5i}$

8 a) $3_{45^\circ} \cdot 2_{15^\circ} = (3\cdot 2)_{45^\circ+15^\circ} = \boxed{6_{60^\circ}}$

$= 6\cos 60^\circ + i(6\,\mathrm{sen}\,60^\circ) = \boxed{3 + 5{,}2i}$

b) $6_{-21^\circ} : 2_{24^\circ} = (6:2)_{-21^\circ-24^\circ} = 3_{-45^\circ} = \boxed{3_{315^\circ}}$

$= 3\cos 315^\circ + i(3\,\mathrm{sen}\,315^\circ) = \boxed{2{,}1 - 2{,}1i}$

c) $(2_{25^\circ})^3 \cdot 3_{15^\circ} = 8_{75^\circ} \cdot 3_{15^\circ} = \boxed{24_{90^\circ}}$

$= 24\cos 90^\circ + i(24\,\mathrm{sen}\,90^\circ) = \boxed{24i}$

d) $(\sqrt{2} - i)^6$

$z = \sqrt{2} - i$

Módulo : $\sqrt{(\sqrt{2})^2 + (-1)^2} = \sqrt{3}$

Argumento : $\alpha = 360^\circ + \mathrm{arctg}\,\frac{-1}{\sqrt{2}} \simeq 360^\circ - 35^\circ = 325^\circ$

$\Big\} \; z = \sqrt{3}_{325^\circ}$

$z^6 = \left(\sqrt{3}_{325^\circ}\right)^6 = 27_{1950^\circ} = \boxed{27_{150^\circ}}$

$= 27\cos 150^\circ + i(27\,\mathrm{sen}\,150^\circ) = \boxed{-23{,}4 + 13{,}5i}$

9 a) $\sqrt[3]{1+i}$

$z = 1 + i$

Módulo: $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Argumento : $\alpha = \mathrm{arctg}\,\frac{1}{1} = 45^\circ$

$\Big\} \; z = \sqrt{2}_{45^\circ}$

$$\sqrt[3]{\sqrt{2}_{45^\circ}} = \boxed{\begin{cases} \left(\sqrt[3]{\sqrt{2}}\right)_{\beta_1} = \left(\sqrt[6]{2}\right)_{15^\circ} \\ \left(\sqrt[3]{\sqrt{2}}\right)_{\beta_2} = \left(\sqrt[6]{2}\right)_{135^\circ} \\ \left(\sqrt[3]{\sqrt{2}}\right)_{\beta_3} = \left(\sqrt[6]{2}\right)_{255^\circ} \end{cases}}$$

$\beta_1 = \frac{45^\circ + 0\cdot 360^\circ}{3} = 15^\circ$

$\beta_2 = \frac{45^\circ + 1\cdot 360^\circ}{3} = 135^\circ$

$\beta_3 = \frac{45^\circ + 2\cdot 360^\circ}{3} = 255^\circ$

b) $\sqrt[3]{-i} = \sqrt[3]{1_{270^\circ}} = \begin{cases} 1_{\beta_1} = 1_{90^\circ} \\ 1_{\beta_2} = 1_{210^\circ} \\ 1_{\beta_3} = 1_{330^\circ} \end{cases}$

$\beta_1 = \dfrac{270^\circ + 0 \cdot 360^\circ}{3} = 90^\circ$

$\beta_2 = \dfrac{270^\circ + 1 \cdot 360^\circ}{3} = 210^\circ$

$\beta_3 = \dfrac{270^\circ + 2 \cdot 360^\circ}{3} = 330^\circ$

c) $\sqrt[6]{-64} = \sqrt[6]{64_{180^\circ}} = \begin{cases} 2_{\beta_1} = 2_{30^\circ} \\ 2_{\beta_2} = 2_{90^\circ} \\ 2_{\beta_3} = 2_{150^\circ} \\ 2_{\beta_4} = 2_{210^\circ} \\ 2_{\beta_5} = 2_{270^\circ} \\ 2_{\beta_6} = 2_{330^\circ} \end{cases}$

$\beta_1 = \dfrac{180^\circ + 0 \cdot 360^\circ}{6} = 30^\circ$

$\beta_2 = \dfrac{180^\circ + 1 \cdot 360^\circ}{6} = 90^\circ$

$\beta_3 = \dfrac{180^\circ + 2 \cdot 360^\circ}{6} = 150^\circ$

$\beta_4 = \dfrac{180^\circ + 3 \cdot 360^\circ}{6} = 210^\circ$

$\beta_5 = \dfrac{180^\circ + 4 \cdot 360^\circ}{6} = 270^\circ$

$\beta_6 = \dfrac{180^\circ + 5 \cdot 360^\circ}{6} = 330^\circ$

d) $\sqrt[3]{-27} = \sqrt[3]{27_{180^\circ}} = \begin{cases} 3_{\beta_1} = 3_{60^\circ} \\ 3_{\beta_2} = 3_{180^\circ} \\ 3_{\beta_3} = 3_{300^\circ} \end{cases}$

$\beta_1 = \dfrac{180^\circ + 0 \cdot 360^\circ}{3} = 60^\circ$

$\beta_2 = \dfrac{180^\circ + 1 \cdot 360^\circ}{3} = 180^\circ$

$\beta_3 = \dfrac{180^\circ + 2 \cdot 360^\circ}{3} = 300^\circ$

10 a) $\sqrt[5]{\dfrac{32}{-i}} = \sqrt[5]{32i} = \sqrt[5]{32_{90^\circ}}$

$z = \dfrac{32}{-i} = \dfrac{32i}{-i \cdot i} = 32i = 32_{90^\circ}$

$\sqrt[5]{32_{90^\circ}} = \begin{cases} 2_{\beta_1} = 2_{18^\circ} \\ 2_{\beta_2} = 2_{90^\circ} \\ 2_{\beta_3} = 2_{162^\circ} \\ 2_{\beta_4} = 2_{234^\circ} \\ 2_{\beta_5} = 2_{306^\circ} \end{cases}$

$\beta_1 = \dfrac{90^\circ + 0 \cdot 360^\circ}{5} = 18^\circ$

$\beta_2 = \dfrac{90^\circ + 1 \cdot 360^\circ}{5} = 90^\circ$

$\beta_3 = \dfrac{90^\circ + 2 \cdot 360^\circ}{5} = 162^\circ$

$\beta_4 = \dfrac{90^\circ + 3 \cdot 360^\circ}{5} = 234^\circ$

$\beta_5 = \dfrac{90^\circ + 4 \cdot 360^\circ}{5} = 306^\circ$

b) Por el ejercicio 2 sabemos que $z = \frac{i^5 - i^{-8}}{\sqrt{2}\, i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} i = 1_{45^\circ}$

$$z^5 = \left(1_{45^\circ}\right)^5 = 1_{5\cdot 45^\circ} = \boxed{1_{225^\circ}}$$

c) $\left(\frac{1+i}{1-i}\right)^5 = i^5 = i^1 = \boxed{i}$

$$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i-1}{1-i+i+1} = \frac{2i}{2} = i$$

d) $\sqrt[4]{-8+8\sqrt{3}i}$

$z = -8+8\sqrt{3}i$

Módulo: $|z| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = 16$

Argumento: $\alpha = 180^\circ + \operatorname{arctg} \frac{8\sqrt{3}}{-8} = 180^\circ - 60^\circ = 120^\circ$

$\Big\}\ z = 16_{120^\circ}$

$$\sqrt[4]{16_{120^\circ}} = \boxed{\begin{cases} z_{\beta_1} = 2_{30^\circ} \\ z_{\beta_2} = 2_{120^\circ} \\ z_{\beta_3} = 2_{210^\circ} \\ z_{\beta_4} = 2_{300^\circ} \end{cases}}$$

$\beta_1 = \frac{120^\circ + 0\cdot 360^\circ}{4} = 30^\circ$

$\beta_2 = \frac{120^\circ + 1\cdot 360^\circ}{4} = 120^\circ$

$\beta_3 = \frac{120^\circ + 2\cdot 360^\circ}{4} = 210^\circ$

$\beta_4 = \frac{120^\circ + 3\cdot 360^\circ}{4} = 300^\circ$