CAPITULO 8

INTEGRACION DE FUNCIONES RACIONALES D SENO Y COSENO

Existen funciones racionales que conllevan formas trigonométricas, reducibles por si a: seno y coseno. Lo conveniente en tales casos es usar las siguientes sustituciones: $z = \tau g\,\frac{x}{2}$, de donde: $x = 2\operatorname{arc}\tau gz$ y $dx = \frac{2dz}{1+z^2}$. Es fácil llegar a verificar que de lo anterior se consigue: $\operatorname{sen} x = \frac{2z}{1+z^2}$ y $\cos x = \frac{1-z^2}{1+z^2}$

EJERCICIOS DESARROLLADOS

8.1.-Encontrar: $\int \frac{dx}{2-\cos x}$

Solución.- La función racional con expresión trigonométrica es: $\frac{1}{2-\cos x}$, y su solución se hace sencilla, usando sustituciones recomendadas, este es:

$z = \tau g\,\frac{x}{2}$, $x = 2\operatorname{arc}\tau gz$, $dx = \frac{2dz}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$ $\therefore$

$$\int \frac{dx}{2-\cos x} = \int \frac{\frac{2dz}{1+z^2}}{2-\frac{1-z^2}{1+z^2}} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2+2z^2-1+z^2}{\cancel{1+z^2}}} = \int \frac{2dz}{3z^2+1} = \int \frac{2dz}{3(z^2+\frac{1}{3})}$$

$$= \frac{2}{3}\int \frac{dz}{z^2+(\sqrt{\frac{1}{3}})^2} = \frac{2}{3}\sqrt{3}\operatorname{arc}\tau g\sqrt{3}z + c$$, recordando que: $z = \tau g\,\frac{x}{2}$, se tiene:

$$= \frac{2}{3}\sqrt{3}\operatorname{arc}\tau g\sqrt{3}\tau g\,\frac{x}{2} + c$$

Respuesta: $\int \frac{dx}{2-\cos x} = \frac{2}{3}\operatorname{arc}\tau g\sqrt{3}\tau g\,\frac{x}{2} + c$

8.2.-Encontrar: $\int \frac{dx}{2-\operatorname{sen} x}$

Solución.- Forma racional: $\frac{1}{2-\operatorname{sen} x}$,

sustituciones: $z = \tau g\,\frac{x}{2}$, $x = 2\operatorname{arc}\tau gz$, $dx = \frac{2dz}{1+z^2}$, $\operatorname{sen} x = \frac{2z}{1+z^2}$ $\therefore$

$$\int \frac{dx}{2-\operatorname{sen} x} = \int \frac{\frac{2dz}{1+z^2}}{2-\frac{2z}{1+z^2}} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2+2z^2-2z}{\cancel{1+z^2}}} = \int \frac{\cancel{2}dz}{\cancel{2}(1+z^2-z)} = \int \frac{dz}{(z^2-z+1)}$$

Ahora bien: $z^2-z+1=(z^2-z+\frac{1}{4})+1-\frac{1}{4}=(z-\frac{1}{2})^2+\frac{3}{4}=(z-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2$

$$\therefore\int\frac{dx}{(z-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}=\frac{1}{\frac{\sqrt{3}}{2}}\operatorname{arc}\tau g\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c=\frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{\frac{2z-1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}}+c$$

$=\frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2z-1}{\sqrt{3}}+c$,recordando que: $z=\tau g\frac{x}{2}$, se tiene:

$$=\frac{2\sqrt{3}}{3}\operatorname{arc}\tau g\frac{2\tau g\frac{x}{2}-1}{\sqrt{3}}+c$$

Respuesta: $\int\frac{dx}{2-\operatorname{sen}x}=\frac{2\sqrt{3}}{3}\operatorname{arc}\tau g\frac{2\tau g\frac{x}{2}-1}{\sqrt{3}}+c$

8.3.-Encontrar: $\int\frac{d\theta}{4-5\cos\theta}$

Solución.- Forma racional: $\frac{1}{4-5\cos\theta}$,

sustituciones: $z=\tau g\frac{\theta}{2}$, $x=2\operatorname{arc}\tau gz$, $dx=\frac{2dz}{1+z^2}$, $\cos x=\frac{1-z^2}{1+z^2}$

$$\therefore\int\frac{dx}{4-5\cos\theta}=\int\frac{\frac{2dz}{1+z^2}}{4-5\left(\frac{1-z^2}{1+z^2}\right)}=\int\frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{4+4z^2-5+5z^2}{\cancel{1+z^2}}}=\int\frac{2dz}{9z^2-1}=\int\frac{2dz}{9(z^2-\frac{1}{9})}$$

$$=\frac{2}{9}\int\frac{dz}{z^2-(\frac{1}{3})^2}=\frac{\cancel{2}}{9}\frac{1}{\cancel{2}(\frac{1}{3})}\ell\eta\left|\frac{z-\frac{1}{3}}{z+\frac{1}{3}}\right|+c=\frac{1}{3}\ell\eta\left|\frac{3z-1}{3z+1}\right|+c$$

Recordando que: $z=\tau g\frac{\theta}{2}$, se tiene: $=\frac{1}{3}\ell\eta\left|\frac{3\tau g\frac{\theta}{2}-1}{3\tau g\frac{\theta}{2}+1}\right|+c$

Respuesta: $\int\frac{d\theta}{4-5\cos\theta}=\frac{1}{3}\ell\eta\left|\frac{3\tau g\frac{\theta}{2}-1}{3\tau g\frac{\theta}{2}+1}\right|+c$

8.4.-Encontrar: $\int\frac{d\theta}{3\cos\theta+4\operatorname{sen}\theta}$

Solución.- usando las sustituciones recomendadas:

$$\int\frac{d\theta}{3\cos\theta+4\operatorname{sen}\theta}=\int\frac{\frac{2dz}{1+z^2}}{3\left(\frac{1-z^2}{1+z^2}\right)+4\left(\frac{2z}{1+z^2}\right)}=\int\frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{3-3z^2+8z}{\cancel{1+z^2}}}$$

$$= \int \frac{2dz}{-3(z^2 - \frac{8}{3}z - 1)} = -\frac{2}{3}\int \frac{dz}{z^2 - \frac{8}{3}z - 1}$$, pero:

$z^2 - \frac{8}{3}z - 1 = (z^2 - \frac{8}{3}z + \frac{16}{9}) - 1 - \frac{16}{9} = (z - \frac{4}{3})^2 - (\frac{5}{3})^2$, luego:

$$= -\frac{2}{3}\int \frac{dz}{(z-\frac{4}{3})^2 - (\frac{5}{3})^2}$$, sea: $w = z - \frac{4}{3}, dw = dz$; de donde:

$$= -\frac{2}{3}\frac{1}{2(\frac{5}{3})}\ell\eta\left|\frac{z-\frac{4}{3}-\frac{5}{3}}{z-\frac{4}{3}+\frac{5}{3}}\right| + c = -\frac{1}{5}\ell\eta\left|\frac{3z-9}{3z+1}\right| + c$$, como: $z = \tau g \frac{\theta}{2}$, se tiene:

$$= -\frac{1}{5}\ell\eta\left|\frac{3\tau g\frac{\theta}{2} - 9}{3\tau g \frac{\theta}{2} + 1}\right| + c$$

Respuesta: $\displaystyle\int \frac{d\theta}{3\cos\theta + 4\,\mathrm{sen}\,\theta} = -\frac{1}{5}\ell\eta\left|\frac{3\tau g\frac{\theta}{2} - 9}{3\tau g \frac{\theta}{2} + 1}\right| + c$

8.5.-Encontrar: $\displaystyle\int \frac{d\theta}{3 + 2\cos\theta + 2\,\mathrm{sen}\,\theta}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{d\theta}{3+2\cos\theta+2\,\mathrm{sen}\,\theta} = \int \frac{\frac{2dz}{1+z^2}}{3 + 2\left(\frac{1-z^2}{1+z^2}\right) + 2\left(\frac{2z}{1+z^2}\right)} = \int \frac{\frac{2dz}{1+z^2}}{3 + \frac{2-2z^2}{1+z^2} + \frac{4z}{1+z^2}}$$

$$= \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{3+3z^2+2-2z^2+4z}{\cancel{1+z^2}}} = \int \frac{2dz}{z^2+4z+5} = \int \frac{2dz}{(z+2)^2+1} = 2\,\mathrm{arc}\,\tau g(z+2) + c$$

Como: $z = \tau g \frac{\theta}{2}$, se tiene: $= 2\,\mathrm{arc}\,\tau g(\tau g\frac{\theta}{2} + 2) + c$

Respuesta: $\displaystyle\int \frac{d\theta}{3+2\cos\theta+2\,\mathrm{sen}\,\theta} = 2\,\mathrm{arc}\,\tau g(\tau g\frac{\theta}{2} + 2) + c$

8.6.-Encontrar: $\displaystyle\int \frac{dx}{\tau g\theta - \mathrm{sen}\,\theta}$

Solución.- Antes de hacer las sustituciones recomendadas, se buscará la equivalencia correspondiente a $\tau g\theta$

$$\tau g\theta = \frac{\mathrm{sen}\,\theta}{\cos\theta} = \frac{\frac{2z}{\cancel{1+z^2}}}{\frac{1-z^2}{\cancel{1+z^2}}} = \frac{2z}{1-z^2}$$, procédase ahora como antes:

$$\int\frac{dx}{\tau g\theta-\operatorname{sen}\theta}=\int\frac{\frac{2dz}{1+z^2}}{\frac{2z}{1-z^2}+\frac{2z}{1+z^2}}=\int\frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2z(1+z^2)-2z(1-z^2)}{(1-z^2)\cancel{(1+z^2)}}}=\int\frac{2(1-z^2)dz}{\cancel{2z}+2z^3\cancel{-2z}+2z^3}$$

$$=\int\frac{(2-2z^2)dz}{4z^3}=\frac{1}{2}\int z^{-3}dz-\frac{1}{2}\int\frac{dz}{z}=-\frac{1}{4z^2}-\frac{1}{2}\ell\eta|z|+c$$

Como: $z=\tau g\,\theta/2$, se tiene: $=-\frac{1}{4}(\operatorname{co}\tau g^2\,\theta/2)-\frac{1}{2}\ell\eta\left|\tau g\,\theta/2\right|+c$

Respuesta: $\int\frac{dx}{\tau g\theta-\operatorname{sen}\theta}=-\frac{1}{4}(\operatorname{co}\tau g^2\,\theta/2)-\frac{1}{2}\ell\eta\left|\tau g\,\theta/2\right|+c$

8.7.-Encontrar: $\int\frac{dx}{2+\operatorname{sen}x}$

Solución.- usando las sustituciones recomendadas:

$$\int\frac{dx}{2+\operatorname{sen}x}=\int\frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}}=\int\frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{2+2z^2+2z}{\cancel{1+z^2}}}=\int\frac{dz}{z^2+z+1}=\int\frac{dz}{(z^2+z+1/4)+3/4}$$

$$=\int\frac{2dz}{(z+1/2)^2+(\sqrt{3}/2)^2}=\frac{1}{\sqrt{3}/2}\operatorname{arc}\tau g\frac{(z+1/2)}{\sqrt{3}/2}+c=\frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2z+1}{\sqrt{3}}+c$$

Como: $z=\tau g\,x/2$, se tiene: $=\frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2\tau g\,x/2+1}{\sqrt{3}}+c$

Respuesta: $\int\frac{dx}{2+\operatorname{sen}x}=\frac{2}{\sqrt{3}}\operatorname{arc}\tau g\frac{2\tau g\,x/2+1}{\sqrt{3}}+c$

8.8.-Encontrar: $\int\frac{\cos xdx}{1+\cos x}$

Solución.-usando las sustituciones recomendadas:

$$\int\frac{\cos xdx}{1+\cos x}=\int\frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{1+\frac{1-z^2}{1+z^2}}=\int\frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2dz}{\cancel{1+z^2}}\right)}{\frac{1\cancel{+z^2}+1\cancel{-z^2}}{\cancel{1+z^2}}}=\int\frac{\cancel{2}(1-z^2)dz}{(1+z^2)\cancel{2}}=\int\frac{(1-z^2)dz}{(1+z^2)}$$

$$=\int\frac{(-z^2+1)dz}{(z^2+1)}=\int\left(-1+\frac{2}{z^2+1}\right)dz=\int dz+2\int\frac{dz}{z^2+1}=-z+2\operatorname{arc}\tau gz+c$$

Como: $z=\tau g\,x/2$, se tiene: $=-\tau g\frac{x}{2}+2\operatorname{arc}\tau g(\tau g\frac{x}{2})+c$

Respuesta: $\int\frac{\cos xdx}{1+\cos x}=-\tau g\frac{x}{2}+x+c$

8.9.-Encontrar: $\int \frac{dx}{1+\operatorname{sen} x+\cos x}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{dx}{1+\operatorname{sen} x+\cos x}=\int \frac{\frac{2dz}{1+z^2}}{1+\left(\frac{2z}{1+z^2}\right)+\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{2dz}{1\cancel{+z^2}+2z+1\cancel{-z^2}}$$

$=\int \frac{2dz}{2z+2}=\int \frac{dz}{z+1}=\ell\eta\left|z+1\right|+c$, como: $z=\tau g\, x/2$, se tiene: $=\ell\eta\left|\tau g\, x/2+1\right|+c$

Respuesta: $\int \frac{dx}{1+\operatorname{sen} x+\cos x}=\ell\eta\left|\tau g\, x/2+1\right|+c$

8.10.-Encontrar: $\int \frac{dx}{\cos x+2\operatorname{sen} x+3}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{dx}{\cos x+2\operatorname{sen} x+3}=\int \frac{\frac{2dz}{1+z^2}}{\left(\frac{1-z^2}{1+z^2}\right)+\left(\frac{4z}{1+z^2}\right)+3}=\int \frac{2dz}{1-z^2+4z+3+3z^2}=\int \frac{2dz}{2z^2+2z+2}$$

$=\int \frac{dz}{z^2+2z+2}=\int \frac{dz}{(z+1)^2+1}=\operatorname{arc}\tau g(z+1)+c$, como: $z=\tau g\, \theta/2$,

Se tiene: $=\operatorname{arc}\tau g(\tau g\, x/2+1)+c$

Respuesta: $\int \frac{dx}{\cos x+2\operatorname{sen} x+3}=\operatorname{arc}\tau g(\tau g\, x/2+1)+c$

8.11.-Encontrar: $\int \frac{\operatorname{sen} xdx}{1+\operatorname{sen}^2 x}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{\operatorname{sen} xdx}{1+\operatorname{sen}^2 x}=\int \frac{\left(\frac{2z}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)^2}\int \frac{\frac{4zdz}{(1+z^2)^2}}{1+\frac{4z^2}{(1+z^2)^2}}=\int \frac{4zdz}{(1+z^2)^2+4z^2}=\int \frac{4zdz}{1+2z^2+z^4+4z^2}$$

$$=\int \frac{4zdz}{z^4+6z^2+1}=\int \frac{4zdz}{(z^4+6z^2+9)-8}=\int \frac{4zdz}{(z^2+3)^2-(\sqrt{8})^2}$$

Sea: $w=z^2+3, dw=2zdz$

$$=2\int \frac{dw}{w^2-(\sqrt{8})^2}=\frac{\cancel{2}}{\cancel{2}\sqrt{8}}\ell\eta\left|\frac{w-\sqrt{8}}{w+\sqrt{8}}\right|+c=\frac{\sqrt{8}}{8}\ell\eta\left|\frac{w-\sqrt{8}}{w+\sqrt{8}}\right|+c=\frac{\sqrt{8}}{8}\ell\eta\left|\frac{z^2+3-\sqrt{8}}{z^2+3+\sqrt{8}}\right|+c$$

Como: $z=\tau g\, \theta/2$, se tiene: $=\frac{\sqrt{2}}{4}\ell\eta\left|\frac{z^2+3-\sqrt{8}}{z^2+3+\sqrt{8}}\right|+c=\frac{\sqrt{2}}{4}\ell\eta\left|\frac{\tau g^2\, x/2+3-2\sqrt{2}}{\tau g^2\, x/2+3+2\sqrt{2}}\right|+c$

Respuesta: $\int \frac{sen\, x dx}{1+sen^2 x} = \frac{\sqrt{2}}{4}\ell\eta\left|\frac{\tau g^2 \frac{x}{2}+3-2\sqrt{2}}{\tau g^2 \frac{x}{2}+3+2\sqrt{2}}\right|+c$

8.12.-Encontrar: $\int \frac{d\theta}{5+4\cos\theta}$

Solución.-usando las sustituciones recomendadas:

$$\int \frac{dx}{5+4\cos\theta} = \int \frac{\frac{2dz}{1+z^2}}{5+4\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2dz}{5+5z^2+4-4z^2} = \int \frac{2dz}{z^2+9} = 2\int \frac{dz}{z^2+3^2}$$

$= \frac{2}{3}\operatorname{arc}\tau g\frac{z}{3}+c$, como: $z=\tau g\frac{\theta}{2}$, se tiene: $=\frac{2}{3}\operatorname{arc}\tau g\frac{\tau g \frac{\theta}{2}}{3}+c$

Respuesta: $\int \frac{d\theta}{5+4\cos\theta} = \frac{2}{3}\operatorname{arc}\tau g\frac{\tau g \frac{\theta}{2}}{3}+c$

8.14.-Encontrar: $\int \frac{dx}{sen\, x+\cos x}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{dx}{sen\, x+\cos x} = \int \frac{\frac{2dz}{1+z^2}}{\left(\frac{2z}{1+z^2}\right)+\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2dz}{2z+1-z^2} = 2\int \frac{dz}{(-z^2+2z+1)}$$

$$= -2\int \frac{dz}{(z^2-2z+1)-2} = -2\int \frac{dz}{(z-1)^2-(\sqrt{2})^2} = -\cancel{2}\frac{1}{\cancel{2}\sqrt{2}}\ell\eta\left|\frac{z-1-\sqrt{2}}{z-1+\sqrt{2}}\right|+c$$

$= -\frac{\sqrt{2}}{2}\ell\eta\left|\frac{z-1-\sqrt{2}}{z-1+\sqrt{2}}\right|+c$, como: $z=\tau g \frac{x}{2}$, se tiene: $= -\frac{\sqrt{2}}{2}\ell\eta\left|\frac{\tau g \frac{x}{2}-1-\sqrt{2}}{\tau g \frac{x}{2}-1+\sqrt{2}}\right|+c$

Respuesta: $\int \frac{dx}{sen\, x+\cos x} = -\frac{\sqrt{2}}{2}\ell\eta\left|\frac{\tau g \frac{x}{2}-1-\sqrt{2}}{\tau g \frac{x}{2}-1+\sqrt{2}}\right|+c$

8.14.-Encontrar: $\int \frac{\sec x dx}{\sec x+2\tau g x-1}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{\sec x dx}{\sec x+2\tau g x-1} = \int \frac{\frac{1}{\cos x}dx}{\frac{1}{\cos x}+\frac{2 sen\, x}{\cos x}-1} = \int \frac{dx}{1+2 sen\, x-\cos x} = \int \frac{\frac{2dz}{1+z^2}}{1+\left(\frac{4z}{1+z^2}\right)-\left(\frac{1-z^2}{1+z^2}\right)}$$

$$= \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{\cancel{1}+z^2+4z\cancel{-1}+z^2}{\cancel{1+z^2}}} = \int \frac{2dz}{2z^2+4z} = \int \frac{\cancel{2}dz}{\cancel{2}(z^2+2z)} = \int \frac{dz}{z(z+2)} \; (*)$$

Ahora bien: $\dfrac{1}{z(z+2)} = \dfrac{A}{z} + \dfrac{B}{z+2}$, de donde:

$\dfrac{1}{\cancel{z(z+2)}} = \dfrac{A(z+2)+B(z)}{\cancel{z(z+2)}} \Rightarrow 1 = A(z+2)+B(z)$, de donde: $A = \frac{1}{2}, B = -\frac{1}{2}$

$$(*) \int \frac{dz}{z(z+2)} = \int \frac{\frac{1}{2}dz}{z} - \int \frac{\frac{1}{2}dz}{z+2} = \frac{1}{2}\int \frac{dz}{z} - \frac{1}{2}\int \frac{dz}{z+2} = \frac{1}{2}\ell\eta|z| - \frac{1}{2}\ell\eta|z+2| + c$$

$= \dfrac{1}{2}\ell\eta\left|\dfrac{z}{z+2}\right| + c$, como: $z = \tau g \frac{x}{2}$, se tiene: $= \dfrac{1}{2}\ell\eta\left|\dfrac{\tau g \frac{x}{2}}{\tau g \frac{x}{2}+2}\right| + c$

Respuesta: $\displaystyle\int \frac{\sec x dx}{\sec x + 2\tau g x - 1} = \frac{1}{2}\ell\eta\left|\frac{\tau g \frac{x}{2}}{\tau g \frac{x}{2}+2}\right| + c$

8.15.-Encontrar: $\displaystyle\int \frac{dx}{1-\cos x + \operatorname{sen} x}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{dx}{1-\cos x+\operatorname{sen} x} = \int \frac{\frac{2dz}{1+z^2}}{1-\left(\frac{1-z^2}{1+z^2}\right)+\left(\frac{2z}{1+z^2}\right)} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{\cancel{1}+z^2\cancel{-1}+z^2+2z}{\cancel{1+z^2}}} = \int \frac{2dz}{2z^2+2z}$$

$$= \int \frac{\cancel{2}dz}{\cancel{2}(z^2+z)} = \int \frac{dz}{z(z+1)} \; (*)$$

Ahora bien: $\dfrac{1}{z(z+1)} = \dfrac{A}{z} + \dfrac{B}{z+1}$, de donde se tiene:

$\dfrac{1}{\cancel{z(z+1)}} = \dfrac{A(z+1)+B(z)}{\cancel{z(z+1)}} \Rightarrow 1 = A(z+1)+B(z)$, de donde: $A=1, B=-1$, luego:

$\displaystyle\int \frac{dz}{z(z+1)} = \int \frac{dz}{z} - \int \frac{dz}{z+1} = \ell\eta|z| - \ell\eta|z+1| + c = \ell\eta\left|\frac{z}{z+1}\right| + c$, como: $z = \tau g \frac{x}{2}$,

Se tiene: $= \ell\eta\left|\dfrac{\tau g \frac{x}{2}}{\tau g \frac{x}{2}+1}\right| + c$

Respuesta: $\displaystyle\int \frac{dx}{1-\cos x+\operatorname{sen} x} = \ell\eta\left|\frac{\tau g \frac{x}{2}}{\tau g \frac{x}{2}+1}\right| + c$

8.16.-Encontrar: $\displaystyle\int \frac{dx}{8-4\operatorname{sen} x+7\cos x}$

Solución.- usando las sustituciones recomendadas:

$$\int \frac{dx}{8-4\operatorname{sen} x+7\cos x}=\int \frac{\frac{2dz}{1+z^2}}{8-\left(\frac{8z}{1+z^2}\right)+7\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{8+8z^2-8z+7-7z^2}{\cancel{1+z^2}}}$$

$$=\int \frac{2dz}{z^2-8z+15}=\int \frac{2dz}{(z-3)(z-5)}\ (*)$$

Ahora bien: $\frac{2}{(z-3)(z-5)}=\frac{A}{(z-3)}+\frac{B}{(z-5)}$, de donde se tiene:

$\Rightarrow 2=A(z-5)+B(z-3)$, de donde: $A=-1, B=1$, luego:

$$\int \frac{2dz}{(z-3)(z-5)}=-\int \frac{dz}{z-3}+\int \frac{dz}{z-5}=-\ell\eta\left|z-3\right|+\ell\eta\left|z-5\right|+c=\ell\eta\left|\frac{z-5}{z-3}\right|+c,$$

como: $z=\tau g\, {}^{x}\!/_{2}$, se tiene: $=\ell\eta\left|\frac{\tau g\, {}^{x}\!/_{2}-5}{\tau g\, {}^{x}\!/_{2}-3}\right|+c$

Respuesta: $\int \frac{dx}{8-4\operatorname{sen} x+7\cos x}=\ell\eta\left|\frac{\tau g\, {}^{x}\!/_{2}-5}{\tau g\, {}^{x}\!/_{2}-3}\right|+c$

EJERCICIOS PROPUESTOS

8.17.- $\int \frac{dx}{1+\cos x}$

8.18.- $\int \frac{dx}{1-\cos x}$

8.19.- $\int \frac{\operatorname{sen} x dx}{1+\cos x}$

8.20.- $\int \frac{\cos x dx}{2-\cos x}$

8.21.- $\int \frac{d\theta}{5-4\cos\theta}$

8.22.- $\int \frac{\operatorname{sen}\theta d\theta}{\cos^2\theta-\cos\theta-2}$

8.23.- $\int \sec x dx$

8.24.- $\int \frac{\cos\theta d\theta}{5+4\cos\theta}$

8.25.- $\int \frac{d\theta}{\cos\theta+\operatorname{co}\tau g\theta}$

RESPUESTAS

8.17.- $\int \frac{dx}{1+\cos x}$

Solución.-

$$\int \frac{dx}{1+\cos x}=\int \frac{\frac{2dz}{1+z^2}}{1+\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{1+z^2+1-z^2}{\cancel{1+z^2}}}=\int dz=z+c=\tau g\, {}^{x}\!/_{2}+c$$

8.18.- $\int \frac{dx}{1-\cos x}$

Solución.-

$$\int \frac{dx}{1-\cos x}=\int \frac{\frac{2dz}{1+z^2}}{1-\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{\cancel{1}+z^2\cancel{-1}-z^2}{\cancel{1+z^2}}}=\int \frac{\cancel{2}dz}{\cancel{2}z^2}=-\frac{1}{z}+c=-\text{co}\,\tau g\, {}^{x}\!/_{2}+c$$

8.19.- $\int \frac{\text{sen}\, x dx}{1+\cos x}$

Solución.-

$$\int \frac{\text{sen}\, x dx}{1+\cos x}=\int \frac{\left(\frac{2z}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{1+\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{\frac{4zdz}{\cancel{(1+z^2)}^2}}{\frac{1+\cancel{z^2}+1\cancel{-z^2}}{\cancel{1+z^2}}}=\int \frac{4zdz}{2(1+z^2)}=\int \frac{2zdz}{(1+z^2)}$$

$$=\ell\eta\left|1+z^2\right|+c=\ell\eta\left|1+\tau g^2\, {}^{x}\!/_{2}\right|+c$$

8.20.- $\int \frac{\cos x dx}{2-\cos x}$

Solución.-

$$\int \frac{\cos x dx}{2-\cos x}=\int\left(-1+\frac{2}{2-\cos x}\right)dx=-\int dx+2\int \frac{dx}{2-\cos x}=-\int dx+2\int \frac{\left(\frac{2dz}{1+z^2}\right)}{2-\left(\frac{1-z^2}{1+z^2}\right)}$$

$$=-\int dx+2\int \frac{\frac{2dz}{\cancel{(1+z^2)}}}{\frac{2+2z^2-1+z^2}{\cancel{1+z^2}}}=-\int dx+2\int \frac{2dz}{3z^2+1}=-\int dx+\frac{4}{3}\int \frac{dz}{(z^2+{}^{1}\!/_{3})}$$

$$=-\int dx+\frac{4}{3}\int \frac{dz}{z^2+({}^{1}\!/_{\sqrt{3}})^2}=-x+{}^{4}\!/_{3}\,\frac{1}{{}^{1}\!/_{\sqrt{3}}}\text{arc}\,\tau g\,\frac{z}{{}^{1}\!/_{\sqrt{3}}}+c=-x+\frac{4\sqrt{3}}{3}\text{arc}\,\tau g\sqrt{3}z+c$$

$$=-x+\frac{4\sqrt{3}}{3}\text{arc}\,\tau g(\sqrt{3}\tau g\, {}^{x}\!/_{2})+c$$

8.21.- $\int \frac{d\theta}{5-4\cos\theta}$

Solución.-

$$\int \frac{d\theta}{5-4\cos\theta}=\int \frac{\left(\frac{2dz}{1+z^2}\right)}{5-4\left(\frac{1-z^2}{1+z^2}\right)}=\int \frac{\frac{2dz}{\cancel{(1+z^2)}}}{\frac{5+5z^2-4+4z^2}{\cancel{1+z^2}}}=\int \frac{2dz}{9z^2+1}=\frac{2}{9}\int \frac{dz}{(z^2+1)}$$

$$=\frac{2}{9}\int \frac{dz}{z^2+({}^{1}\!/_{3})^2}=\frac{2}{9}\,\frac{1}{{}^{1}\!/_{3}}\text{arc}\,\tau g\,\frac{z}{{}^{1}\!/_{3}}+c=\frac{2}{3}\text{arc}\,\tau g 3z+c=\frac{2}{3}\text{arc}\,\tau g(3\tau g\, {}^{x}\!/_{2})+c$$

8.22.- $\int \frac{\operatorname{sen}\theta d\theta}{\cos^2\theta - \cos\theta - 2}$

Solución.-

$$\int \frac{\operatorname{sen}\theta d\theta}{\cos^2\theta-\cos\theta-2} = \int \frac{\left(\frac{2z}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)^2 - \left(\frac{1-z^2}{1+z^2}\right) - 2} = \int \frac{\frac{4zdz}{\cancel{(1+z^2)^2}}}{\frac{(1-z^2)^2-(1-z^2)(1+z^2)-2(1+z^2)^2}{\cancel{(1+z^2)^2}}}$$

$$= \int \frac{4zdz}{-6z^2-2} = -\frac{1}{3}\int \frac{2zdz}{(z^2-\frac{1}{3})} = -\frac{1}{3}\ell\eta\left|z^2-\frac{1}{3}\right| + c = -\frac{1}{3}\ell\eta\left|\tau g^2 \frac{x}{2} - \frac{1}{3}\right| + c$$

8.23.- $\int \sec x dx$

Solución.-

$$\int \sec x dx = \int \frac{dx}{\cos x} = \int \frac{\frac{2dz}{\cancel{1+z^2}}}{\frac{1-z^2}{\cancel{1+z^2}}} = \int \frac{2dz}{(1-z^2)} = \int \frac{2dz}{(1+z)(1-z)} \ (*)$$

Ahora bien: $\frac{2}{(1+z)(1-z)} = \frac{A}{1+z} + \frac{B}{1-z}$, de donde: $A=1, B=1$, luego:

$$(*) \int \frac{2dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} - \int \frac{dz}{1-z} = \ell\eta|1+z| - \ell\eta|1-z| + c = \ell\eta\left|\frac{1+z}{1-z}\right| + c$$

Como: $z = \tau g \frac{x}{2}$, Se tiene: $= \ell\eta\left|\frac{1+\tau g \frac{x}{2}}{1-\tau g \frac{x}{2}}\right| + c$

8.24.- $\int \frac{\cos\theta d\theta}{5+4\cos\theta}$

Solución.-

$$\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2dz}{1+z^2}\right)}{5+4\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{\frac{2(1-z^2)dz}{(1+z^2)^{\cancel{2}}}}{\frac{(5+5z^2+4-4z^2)}{\cancel{(1+z^2)}}} = \int \frac{(2-2z^2)dz}{(1+z^2)(9+z^2)}$$

Ahora bien: $\frac{2-2z^2}{(z^2+1)(z^2+9)} = \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+9}$, de donde: $A=0, B=\frac{1}{2}, C=0, D=-\frac{5}{2}$, luego:

$$\int \frac{(2-2z^2)}{(z^2+1)(z^2+9)} = \frac{1}{2}\int \frac{dz}{z^2+1} - \frac{5}{2}\int \frac{dz}{z^2+9} = \frac{1}{2}\operatorname{arc}\tau g z + \frac{5}{2}\operatorname{arc}\tau g \frac{z}{3} + c$$

$$= \frac{1}{2}\operatorname{arc}\tau g \frac{\theta}{2} - \frac{5}{6}\operatorname{arc}\tau g\left(\frac{\tau g \frac{\theta}{2}}{3}\right) + c = \frac{\theta}{4} - \frac{5}{6}\operatorname{arc}\tau g\left(\frac{\tau g \frac{\theta}{2}}{3}\right) + c$$

8.25.- $\int \frac{d\theta}{\cos\theta + \operatorname{co}\tau g\theta}$

Solución.-

$$\int \frac{d\theta}{\cos\theta + \text{co}\,\tau g\theta} = \int \frac{\left(\frac{2dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)+\left(\frac{1-z^2}{2z}\right)} = \int \frac{\frac{2dz}{\cancel{(1+z^2)}}}{\frac{2z(1-z^2)+(1-z^2)(1+z^2)}{\cancel{(1+z^2)}2z}}$$

$$= \int \frac{4zdz}{2z(1-z^2)+(1-z^2)(1+z^2)} = \int \frac{4zdz}{(1-z^2)(z^2+2z+1)} = \int \frac{4zdz}{(1+z^3)(1-z)}(*)$$

Ahora bien: $\frac{4z}{(1+z^3)(1-z)} = \frac{A}{1+z} + \frac{B}{(1+z)^2} + \frac{C}{(1+z)^3} + \frac{D}{(1-z)}$

De donde: $A = 1/2, B = 1, C = -2, D = 1/2$, luego:

$$(*)\int \frac{4z}{(1+z^3)(1-z)} = \frac{1}{2}\int \frac{dz}{1+z} + \int \frac{dz}{(1+z)^2} - 2\int \frac{dz}{(1+z)^3} + \frac{1}{2}\int \frac{dz}{1-z}$$

$$= \frac{1}{2}\ell\eta|1+z| - \frac{1}{1+z} + \frac{1}{(1+z)^2} - \frac{1}{2}\ell\eta|1-z| + c = \frac{1}{2}\ell\eta\left|\frac{1+z}{1-z}\right| - \frac{1}{1+z} + \frac{1}{(1+z)^2} + c$$

$$= \frac{1}{2}\ell\eta\left|\frac{1+z}{1-z}\right| + \frac{-(1+z)+1}{(1+z)^2} + c = \frac{1}{2}\ell\eta\left|\frac{1+z}{1-z}\right| - \frac{z}{(1+z)^2} + c = \frac{1}{2}\ell\eta\left|\frac{1+\tau g\,\theta/2}{1-\tau g\,\theta/2}\right| - \frac{\tau g\,\theta/2}{(1+\tau g\,\theta/2)^2} + c$$